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Generalizations of Stampacchia Lemma and applications to quasilinear elliptic systems

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ABSTRACT

We present two generalizations of the classical Stampacchia Lemma. As applications, we consider Dirichlet problem for elliptic systems of the form

$$\begin{cases} -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\sum_{\beta=1}^{N}\sum_{j=1}^{n} a_{i,j}^{\alpha,\beta}(x,u(x))\frac{\partial u^{\beta}(x)}{\partial x_j}\right) = f^{\alpha}, & \text{in } \Omega, \\ u(x)=0, & \text{on } \partial\Omega. \end{cases}$$

Under ellipticity and degenerate ellipticity conditions of the diagonal coefficients and proportional conditions of the off-diagonal coefficients, we derive some global regularity results.

1. Two generalizations of Stampacchia Lemma

Guido Stampacchia (1922–1978), an Italian mathematician, was famous for his works on the theory of variational inequalities, the calculus of variation and the theory of elliptic PDEs. Among such works, there is the following Stampacchia Lemma (see Lemma 4.1 in [1]):

Lemma 1.1. *Let c_1, α, β be positive constants. Let $\varphi : [k_0, +\infty) \to [0, +\infty)$ be nonincreasing and such that*

$$\varphi(h) \le \frac{c_1}{(h-k)^{\alpha}}[\varphi(k)]^{\beta} \tag{1.1}$$

for every h, k with $h > k \ge k_0$. It results that:
(i) *if $\beta > 1$ then we have*

$$\varphi(k_0 + d) = 0,$$

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where

$$d^\alpha = c_1[\varphi(k_0)]^{\beta-1}2^{\frac{\alpha\beta}{\beta-1}}.$$

(ii) *if* $\beta = 1$ *then for any* $k \geq k_0$ *we have*

$$\varphi(k) \leq \varphi(k_0)e^{1-(c_1 e)^{-\frac{1}{\alpha}}(k-k_0)}.$$

(iii) *if* $\beta < 1$ *and* $k_0 > 0$ *then for any* $k \geq k_0$ *we have*

$$\varphi(k) \leq 2^{\frac{\alpha}{(1-\beta)^2}} \left\{ c_1^{\frac{1}{1-\beta}} + (2k_0)^{\frac{\alpha}{1-\beta}}\varphi(k_0) \right\} \left(\frac{1}{k}\right)^{\frac{\alpha}{1-\beta}}.$$

Stampacchia Lemma is an efficient tool in dealing with regularity issues of solutions of elliptic PDEs as well as minima of variational integrals, and is used till now repeatedly by many mathematicians. As an example, we consider solutions of the following linear Dirichlet problem (see [2]):

$$\begin{cases} -\mathrm{div}(M(x)Du(x)) = g(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$, $n > 2$, $M(x) : \Omega \to \mathbb{R}^{n\times n}$ is a matrix satisfying

$$M(x)\xi\cdot\xi \geq \alpha|\xi|^2 \quad \text{and} \quad |M(x)| \leq \beta, \quad \text{a.e. } \Omega$$

for some $0 < \alpha \leq \beta < \infty$, and $g \in L^m(\Omega)$ with $m > (2^*)' = \frac{2n}{n+2}$. We use

$$G_k(u) = u - T_k(u) = u - \min\left\{1, \frac{k}{|u|}\right\}u$$

as test function and we have

$$\alpha\int_\Omega |DG_k(u)|^2 dx \leq \int_\Omega gG_k(u)dx.$$

The use of Hölder inequality and Sobolev inequality yield for any $h > k \geq 0$,

$$|A_h| \leq c_g \frac{|A_k|^{2^*\left(\frac{1}{(2^*)'}-\frac{1}{m}\right)}}{(h-k)^{2^*}},$$

where

$$A_k = \{x \in \Omega : |u(x)| > k\}$$

is the superlevel set of u, and $|A_k|$ denotes the Lebesgue measure of A_k. The condition (1.1) holds with

$$k_0 = 0,\ \varphi(k) = |A_k|,\ c_1 = c_g,\ \alpha = 2^*,\ \beta = 2^*\left(\frac{1}{(2^*)'} - \frac{1}{m}\right),$$

then one can use Stampacchia Lemma (Lemma 1.1) to derive that:

- if $2^*\left(\frac{1}{(2^*)'} - \frac{1}{m}\right) > 1$ (that is $m > \frac{n}{2}$), then there exists $d > 0$ such that $|A_d| = 0$: u is bounded ($|u(x)| \leq d$ a.e. Ω);
- if $2^*\left(\frac{1}{(2^*)'} - \frac{1}{m}\right) = 1$ (that is $m = \frac{n}{2}$), then there exists $\lambda > 0$ such that $e^{\lambda|u|} \in L^1(\Omega)$: u belongs to the exponential class $\mathrm{Exp}(\Omega)$;
- if $0 < 2^*\left(\frac{1}{(2^*)'} - \frac{1}{m}\right) < 1$ (that is $(2^*)' < m < \frac{n}{2}$), then there exists $\tilde{c}_0 > 0$ such that $|A_k| \leq \tilde{c}_0\left(\frac{1}{k}\right)^{\frac{\alpha}{1-\beta}} = \tilde{c}_0\left(\frac{1}{k}\right)^{m^{**}}$: u belongs to the Marcinkiewicz space $L^{m^{**}}_{weak}(\Omega)$, $m^{**} = \frac{nm}{n-2m}$.

The above example illustrates how Stampacchia Lemma can be used to derive regularity results. For some other results related to Lemma 1.1, we refer to [3–13].

As far as we know, the first (incomplete) version of Stampacchia Lemma was given in 1960 in [14]. There are different versions of the Stampacchia Lemma, see, for example, [7,9,15]. In [7], the authors compared two versions of Stampacchia Lemma: in (1.1) we take $h = 2k$ and a constant c_2 in place of c_1,

$$\varphi(2k) \leq \frac{c_2}{k^\alpha}[\varphi(k)]^\beta, \quad \forall k \geq k_0. \tag{1.2}$$

It is obvious that (1.1) implies (1.2) with $c_2 = c_1$. In [7], the authors proved that (see Remarks 1, 2, 3 in [7]):

- for the case $0 < \beta < 1$, the two assumptions (1.1) and (1.2) are equivalent;
- for the case $\beta = 1$, (1.1) is stronger than (1.2). More precisely, the function $\varphi(k) = e^{-(\ln k)^2}$, $k \geq 1$, verifies (1.2) with $\beta = 1$, $\alpha = 2\ln 2$, $c_2 = 2^{-\ln 2}$ but it does not satisfy (1.1), for any choice of the two constants $\alpha > 0$ and $c_1 > 0$;
- for the case $\beta > 1$, (1.1) is also stronger than (1.2). More precisely, the function $\varphi(k) = e^{-k^p}$, $p = \log_2(2\beta)$, $k \geq 1$, verifies (1.2) with $\beta > 1$, $c_2 = 1$, any $\alpha > 0$ and a suitable $k_0 = k_0(\alpha, \beta) \geq 1$, but it does not satisfy (1.1) for any choice of the three constants $\beta > 1$, $\alpha > 0$ and $c_1 > 0$.

Due to the importance of the Stampacchia lemma (Lemma 1.1) in the regularity theory of partial differential equations, we now give two generalizations.

1.1. The first generalization.

The first generalization is the following lemma, which can be used in dealing with regularity issues of elliptic systems, see Section 2.1.

Lemma 1.2. *Let c_3, α, β be positive constants, $k_0 > 0$ and $N > 1$. Let $\varphi : [k_0, +\infty) \to [0, +\infty)$ be nonincreasing and such that*

$$\varphi(Nh) \leq \frac{c_3}{(h-k)^\alpha}[\varphi(k)]^\beta \tag{1.3}$$

for every h, k with $h > k \geq k_0$. It results that:

(I) *if $\beta < 1$ then for any $k \geq k_0$ we have*

$$\varphi(k) \leq c_4 \left(\frac{1}{k}\right)^{\frac{\alpha}{1-\beta}}, \tag{1.4}$$

where

$$c_4 = \max\left\{ (2N)^{\frac{\alpha(2-\beta)}{(1-\beta)^2}} \left[1 + \varphi(k_0)\left(\frac{(k_0)^\alpha}{c_3}\right)^{1/(1-\beta)}\right]^\beta c_3^{\frac{1}{1-\beta}}, \varphi(k_0)(2Nk_0)^{\frac{\alpha}{1-\beta}} \right\}.$$

(II) *if $\beta = 1$ then for any $k \geq k_0$ and any $q > 1$, we have*

$$\varphi(k) \leq c_5 \left(\frac{1}{k}\right)^{\tilde{q}}, \tag{1.5}$$

where

$$\tilde{q} = \max\{q, 1 + \alpha\}$$

and

$$c_5 = \max\left\{ (2N)^{\frac{\tilde{q}(\tilde{q}+\alpha)}{\alpha}} \left[1 + \varphi(k_0)\left(\frac{(k_0)^\alpha}{c_3[1+\varphi(k_0)]^{\frac{\alpha}{\tilde{q}}}}\right)^{\frac{\tilde{q}}{\alpha}}\right]^{1-\frac{\alpha}{\tilde{q}}} \left(c_3[1+\varphi(k_0)]^{\frac{\alpha}{\tilde{q}}}\right)^{\frac{\tilde{q}}{\alpha}}, \varphi(k_0)(2Nk_0)^{\tilde{q}} \right\}.$$

(III) *if $\beta > 1$ then we have two cases: there exists $k \geq k_0$ such that $\varphi(k) = 0$, or for any $k \geq k_0$ we have $\varphi(k) > 0$. In such a second case, for any $k \geq \tilde{k}_0$,*

$$\varphi(k) \leq \max\left\{1, \varphi(\tilde{k}_0) c_6^{N^2(\tilde{k}_0+2c_3^{1/\alpha})\log_N\sqrt{\beta}}\right\} c_6^{-k^{\log_N\sqrt{\beta}}},$$

where

$$c_6 = c_7^{\frac{1}{\beta^{\log_N\sqrt{Nc_8}}}}, \quad c_7 = N^{\frac{\alpha}{\beta^2}}\varphi(\tilde{k}_0)^{-1}, \quad c_8 = \max\left\{\frac{\tilde{k}_0 + 2c_3^{1/\alpha}}{N}, \frac{c_3^{1/\alpha}}{N\ln N}, 1\right\},$$

and $\tilde{k}_0 \geq k_0$ is a constant such that $c_7 > 1$, see Remark 1.2.

Proof (I). $\beta < 1$. We take $h = 2k$ in (1.3) and we have

$$\varphi(2Nk) \leq \frac{c_3}{k^\alpha}[\varphi(k)]^\beta, \; N > 1, k \geq k_0. \tag{1.6}$$

Let

$$\varphi_1(s) = \varphi(s)\left(\frac{s^\alpha}{c_3}\right)^{\frac{1}{1-\beta}}. \tag{1.7}$$

(1.7) together with (1.6) implies

$$\begin{aligned}
\varphi_1(2Nk) &= \varphi(2Nk)\left(\frac{(2Nk)^\alpha}{c_3}\right)^{\frac{1}{1-\beta}} \\
&\leq \frac{c_3}{k^\alpha}[\varphi(k)]^\beta (2N)^{\frac{\alpha}{1-\beta}}\left(\frac{k^\alpha}{c_3}\right)^{\frac{1}{1-\beta}} \\
&= (2N)^{\frac{\alpha}{1-\beta}}[\varphi_1(k)]^\beta,
\end{aligned}$$

from which we derive, for any positive integer $j \in \mathbb{N}^+$,

$$\begin{aligned}
& \varphi_1\left((2N)^j k_0\right) \\
= \; & \varphi_1\left(2N\left((2N)^{j-1}k_0\right)\right) \\
\leq \; & (2N)^{\frac{\alpha}{1-\beta}}\left[\varphi_1((2N)^{j-1}k_0)\right]^\beta \\
\leq \; & (2N)^{\frac{\alpha}{1-\beta}}\left\{(2N)^{\frac{\alpha}{1-\beta}}\left[\varphi_1\left((2N)^{j-2}k_0\right)\right]^\beta\right\}^\beta \\
= \; & (2N)^{\frac{\alpha}{1-\beta}}(2N)^{\frac{\alpha\beta}{1-\beta}}\left[\varphi_1\left((2N)^{j-2}k_0\right)\right]^{\beta^2} \\
\leq \; & \cdots\cdots \\
\leq \; & (2N)^{\frac{\alpha}{1-\beta}}(2N)^{\frac{\alpha\beta}{1-\beta}}\cdots(2N)^{\frac{\alpha\beta^{j-1}}{1-\beta}}\left[\varphi_1(k_0)\right]^{\beta^j} \\
\leq \; & (2N)^{\frac{\alpha}{(1-\beta)^2}}\left[1+\varphi_1(k_0)\right]^\beta,
\end{aligned} \tag{1.8}$$

where we have used the facts

$$\frac{\alpha}{1-\beta} + \frac{\alpha\beta}{1-\beta} + \cdots + \frac{\alpha\beta^{j-1}}{1-\beta} = \frac{\alpha}{1-\beta}\left(1+\beta+\cdots+\beta^{j-1}\right) \leq \frac{\alpha}{(1-\beta)^2}$$

and

$$[\varphi_1(k_0)]^{\beta^j} \leq [1+\varphi_1(k_0)]^{\beta^j} \leq [1+\varphi_1(k_0)]^\beta.$$

For any $k \geq k_0$, we split the proof into two cases: $k \geq 2Nk_0$ and $k_0 \leq k < 2Nk_0$:
(1) for the first case $k \geq 2Nk_0$, there exists a positive integer $j \in \mathbb{N}^+$ such that

$$(2N)^j k_0 \leq k < (2N)^{j+1}k_0.$$

By virtue of (1.7), (1.8) and the nonincreasing property of $\varphi(k)$, one has

$$\begin{aligned} \varphi_1(k) &= \varphi(k)\left(\frac{k^\alpha}{c_3}\right)^{\frac{1}{1-\beta}} \\ &\le \varphi((2N)^j k_0)\left(\frac{\left((2N)^{j+1}k_0\right)^\alpha}{c_3}\right)^{\frac{1}{1-\beta}} \\ &= (2N)^{\frac{\alpha}{1-\beta}}\varphi_1\left((2N)^j k_0\right) \\ &\le (2N)^{\frac{\alpha}{1-\beta}}(2N)^{\frac{\alpha}{(1-\beta)^2}}\left[1+\varphi_1(k_0)\right]^\beta \\ &= (2N)^{\frac{\alpha(2-\beta)}{(1-\beta)^2}}\left[1+\varphi_1(k_0)\right]^\beta . \end{aligned} \tag{1.9}$$

Substituting (1.9) into (1.7) we arrive at

$$\varphi(k) = \varphi_1(k)\left(\frac{c_3}{k^\alpha}\right)^{\frac{1}{1-\beta}} \le (2N)^{\frac{\alpha(2-\beta)}{(1-\beta)^2}}[1+\varphi_1(k_0)]^\beta c_3^{\frac{1}{1-\beta}}\left(\frac{1}{k}\right)^{\frac{\alpha}{1-\beta}} . \tag{1.10}$$

(2) for the second case $k_0 \le k < 2Nk_0$, one has

$$\varphi(k) \le \varphi(k_0) = \varphi(k_0)\frac{k^{\frac{\alpha}{1-\beta}}}{k^{\frac{\alpha}{1-\beta}}} \le \varphi(k_0)(2Nk_0)^{\frac{\alpha}{1-\beta}}\left(\frac{1}{k}\right)^{\frac{\alpha}{1-\beta}} . \tag{1.11}$$

(1.11) together with (1.10) yields the desired result (1.4).

(**II**) $\beta = 1$. For any $k \ge k_0$ and any $q > 1$, we let $\tilde{q} = \max\{q, 1+\alpha\}$. (1.3) implies

$$\begin{aligned} \varphi(Nh) &\le \frac{c_3}{(h-k)^\alpha}\varphi(k) \\ &= \frac{c_3}{(h-k)^\alpha}[\varphi(k)]^{\frac{\alpha}{\tilde{q}}}[\varphi(k)]^{1-\frac{\alpha}{\tilde{q}}} \\ &\le \frac{c_3[\varphi(k_0)]^{\frac{\alpha}{\tilde{q}}}}{(h-k)^\alpha}[\varphi(k)]^{1-\frac{\alpha}{\tilde{q}}} \\ &\le \frac{c_3[1+\varphi(k_0)]^{\frac{\alpha}{q}}}{(h-k)^\alpha}[\varphi(k)]^{1-\frac{\alpha}{\tilde{q}}}. \end{aligned}$$

Since $0 < 1-\frac{\alpha}{\tilde{q}} < 1$, then one can make use of (1.4) to derive (1.5) (with $c_3[1+\varphi(k_0)]^{\alpha/q}$ in place of c_3 and $1-\frac{\alpha}{\tilde{q}}$ in place of β).

(**III**) $\beta > 1$. For any nonnegative integer $s \ge 0$, we let

$$k = N^s\left(\tilde{k}_0 + c_3^{1/\alpha}s\right) = \tilde{k}_s$$

and

$$Nh = N^{s+1}\left(\tilde{k}_0 + c_3^{1/\alpha}(s+1)\right) = \tilde{k}_{s+1},$$

where we recall that $\tilde{k}_0 \ge k_0$ is a constant such that $c_7 = N^{\frac{\alpha}{\beta^2}}\varphi(\tilde{k}_0)^{-1} > 1$. It is obvious that

$$h = N^s\left(\tilde{k}_0 + c_3^{1/\alpha}(s+1)\right) > k,$$

which allows us to take k and h as above in (1.3) and we have

$$\varphi(\tilde{k}_{s+1}) \le \frac{c_3}{(N^s c_3^{1/\alpha})^\alpha}[\varphi(\tilde{k}_s)]^\beta = \frac{1}{N^{s\alpha}}[\varphi(\tilde{k}_s)]^\beta. \tag{1.12}$$

Note that the above inequality holds true for the case $s = 0$, that is,

$$\varphi(\tilde{k}_1) \le [\varphi(\tilde{k}_0)]^\beta. \tag{1.13}$$

Thus, for any positive integer $s \geq 1$,

$$\begin{aligned}
& \varphi(\tilde{k}_{s+1}) \\
\leq\ & \frac{1}{N^{s\alpha}}[\varphi(\tilde{k}_s)]^{\beta} \\
\leq\ & \frac{1}{N^{s\alpha}}\left[\frac{1}{N^{(s-1)\alpha}}\left[\varphi(\tilde{k}_{s-1})\right]^{\beta}\right]^{\beta} \\
=\ & \frac{1}{N^{s\alpha}}\frac{1}{N^{(s-1)\alpha\beta}}\left[\varphi(\tilde{k}_{s-1})\right]^{\beta^2} \\
\leq\ & \cdots\cdots \\
\leq\ & \frac{1}{N^{s\alpha}N^{(s-1)\alpha\beta}N^{(s-2)\alpha\beta^2}\cdots N^{1\cdot\alpha\beta^{s-1}}}[\varphi(\tilde{k}_1)]^{\beta^s} \\
\leq\ & \frac{1}{N^{s\alpha}N^{(s-1)\alpha\beta}N^{(s-2)\alpha\beta^2}\cdots N^{1\cdot\alpha\beta^{s-1}}}[\varphi(\tilde{k}_0)]^{\beta^{s+1}} \\
\leq\ & \frac{1}{N^{\alpha\beta^{s-1}}}[\varphi(\tilde{k}_0)]^{\beta^{s+1}} \\
=\ & \left(N^{\frac{\alpha}{\beta^2}}\varphi(\tilde{k}_0)^{-1}\right)^{-\beta^{s+1}},
\end{aligned} \tag{1.14}$$

where we have used (1.12), (1.13) and the fact

$$N > 1, \alpha > 0, s \geq 1 \Rightarrow N^{s\alpha}N^{(s-1)\alpha\beta}N^{(s-2)\alpha\beta^2}\cdots N^{1\cdot\alpha\beta^{s-1}} \geq N^{\alpha\beta^{s-1}}.$$

(1.14) yields that, for any positive integer $s \geq 2$,

$$\varphi(\tilde{k}_s) \leq \left(N^{\frac{\alpha}{\beta^2}}\varphi(\tilde{k}_0)^{-1}\right)^{-\beta^s}. \tag{1.15}$$

For any $k \geq \tilde{k}_0$, we distinguish between two cases: $k \geq \tilde{k}_2$ and $\tilde{k}_0 \leq k < \tilde{k}_2$.

For the case $k \geq \tilde{k}_2$, there exists a positive integer $s \geq 2$ such that

$$\tilde{k}_s \leq k < \tilde{k}_{s+1}.$$

(1.15) allows us to estimate

$$\varphi(k) \leq \varphi(\tilde{k}_s) \leq \left(N^{\frac{\alpha}{\beta^2}}\varphi(\tilde{k}_0)^{-1}\right)^{-\beta^s}. \tag{1.16}$$

We let

$$c_8 = \max\left\{\frac{\tilde{k}_0 + 2c_3^{1/\alpha}}{N}, \frac{c_3^{1/\alpha}}{N\ln N}, 1\right\},$$

then one can prove that, for $1 \leq s \in \mathbb{R}$,

$$\tilde{k}_0 + c_3^{1/\alpha}(s+1) \leq c_8 N^s. \tag{1.17}$$

In fact, let

$$f(s) = c_8 N^s - \left[\tilde{k}_0 + c_3^{1/\alpha}(s+1)\right], \ s \geq 1.$$

It is obvious that

$$f(1) = c_8 N - \left[\tilde{k}_0 + 2c_3^{1/\alpha}\right] \geq 0,$$

and

$$f'(s) = c_8 N^s \ln N - c_3^{1/\alpha} \geq c_8 N \ln N - c_3^{1/\alpha} \geq 0,$$

so for all $s \geq 1$, $f(s) \geq 0$. (1.17) is proved.

From (1.17) and the fact $k < \tilde{k}_{s+1}$ we know that

$$k < N^{s+1}\left(\tilde{k}_0 + c_3^{1/\alpha}(s+1)\right) \leq c_8 N^{2s+1},$$

which yields

$$\begin{aligned} s &> \frac{1}{2}\left[\log_N\left(\frac{k}{c_8}\right)-1\right] \\ &= \frac{1}{2}\left[\log_N k - \log_N c_8 - 1\right] \\ &= \log_N \beta \log_\beta k^{\frac{1}{2}} - \frac{1}{2}\log_N(Nc_8) \\ &= \log_\beta k^{\frac{1}{2}\log_N \beta} - \log_N \sqrt{Nc_8} \\ &= \log_\beta k^{\log_N \sqrt{\beta}} - \log_N \sqrt{Nc_8}. \end{aligned} \tag{1.18}$$

Combining (1.16) and (1.18) we arrive at

$$\begin{aligned} \varphi(k) &\le \left(N^{\frac{\alpha}{\beta^2}}\varphi(\tilde{k}_0)^{-1}\right)^{-\beta^s} = c_7^{-\beta^s} \\ &\le c_7^{-\beta^{\left[\log_\beta k^{\log_N\sqrt{\beta}} - \log_N\sqrt{Nc_8}\right]}} \\ &= c_7^{-\frac{\beta^{\log_\beta k^{\log_N\sqrt{\beta}}}}{\beta^{\log_N\sqrt{Nc_8}}}} \\ &= c_6^{-\beta^{\log_\beta k^{\log_N\sqrt{\beta}}}} \\ &= c_6^{-k^{\log_N\sqrt{\beta}}}, \end{aligned} \tag{1.19}$$

where $c_6 = c_7^{\frac{1}{\beta^{\log_N\sqrt{Nc_8}}}}$.

For the case $\tilde{k}_0 \le k < \tilde{k}_2$ we have

$$\begin{aligned} &\varphi(k) \le \varphi(\tilde{k}_0) = \varphi(\tilde{k}_0)c_6^{\tilde{k}_2\log_N\sqrt{\beta}}c_6^{-\tilde{k}_2\log_N\sqrt{\beta}} \\ &\le \varphi(\tilde{k}_0)c_6^{\tilde{k}_2\log_N\sqrt{\beta}}c_6^{-k\log_N\sqrt{\beta}} \end{aligned} \tag{1.20}$$

(1.19) and (1.20) yield the desired result. □

Two Remarks:

Remark 1.1. We notice that, in the proof of Lemma 1.2-(I), we have taken $h = 2k$ in (1.3) and we have (1.6). We remark that, in the case $0 < \beta < 1$, the two assumptions (1.3) and

$$\varphi(2Nk) \le \frac{\tilde{c}_3}{k^\alpha}[\varphi(k)]^\beta, \ N > 1, k \ge k_0 \tag{$*$}$$

are equivalent. In fact,

(1.3)⇒($*$). We take $h = 2k$ in (1.3) and we get ($*$) with $\tilde{c}_3 = c_3$.

($*$)⇒(1.3). Let us consider $h > k \ge k_0$. We split the proof into two cases: $(2N)^{j+1}k \ge h > (2N)^j k$ for some integer $j \ge 1$ and $2Nk \ge h > k$. For the first case $(2N)^{j+1}k \ge h > (2N)^j k$, since φ non-increases, we have

$$\varphi(h) \le \varphi((2N)^j k) = \varphi\left((2N)(2N)^{j-1}k\right).$$

We keep in mind that $j \ge 1$ so $(2N)^{j-1}k \ge k \ge k_0$ and we can use ($*$) with $(2N)^{j-1}k$ in place of k:

$$\varphi\left((2N)(2N)^{j-1}k\right) \le \frac{\tilde{c}_3}{((2N)^{j-1}k)^\alpha}[\varphi((2N)^{j-1}k)]^\beta.$$

Since $(2N)^{j-1}k \ge k$, we use the monotonicity of φ to have

$$\varphi((2N)^{j-1}k) \le \varphi(k),$$

then

$$[\varphi((2N)^{j-1}k)]^{\beta} \leq [\varphi(k)]^{\beta}.$$

Since $(2N)^{j+1}k \geq h$, we have $((2N)^{j+1}-1)k \geq h-k$, then

$$(2N)^{j-1}k = \frac{(2N)^{j+1}k}{(2N)^2} \geq \frac{((2N)^{j+1}-1)k}{(2N)^2} \geq \frac{h-k}{(2N)^2},$$

thus

$$\begin{aligned} \varphi(h) &\leq \varphi((2N)^j k) = \varphi((2N)(2N)^{j-1}k) \\ &\leq \frac{\tilde{c}_3}{((2N)^{j-1}k)^{\alpha}}[\varphi((2N)^{j-1}k)]^{\beta} \\ &\leq \frac{(2N)^{2\alpha}\tilde{c}_3}{(h-k)^{\alpha}}[\varphi(k)]^{\beta}. \end{aligned}$$

For the second case $2Nk \geq h > k$, since φ non-increases we have

$$\varphi(h) \leq \varphi(k) = [\varphi(k)]^{\beta}[\varphi(k)]^{1-\beta}.$$

The proof of Lemma 1.2-(I) used only (1.6), that is $(*)$, so we get (1.4)

$$\varphi(k) \leq c_4 \left(\frac{1}{k}\right)^{\frac{\alpha}{1-\beta}},$$

where c_4 is defined with $\tilde{c}_3$ in place of c_3. Then

$$\begin{aligned} \varphi(h) &\leq \varphi(k) = [\varphi(k)]^{\beta}[\varphi(k)]^{1-\beta} \\ &\leq [\varphi(k)]^{\beta}\left[c_4\left(\frac{1}{k}\right)^{\frac{\alpha}{1-\beta}}\right]^{1-\beta} \\ &= [\varphi(k)]^{\beta} c_4^{1-\beta}\left(\frac{1}{k}\right)^{\alpha}. \end{aligned}$$

Since $2Nk \geq h$ we get $(2N-1)k \geq h-k$ and $\left(\frac{1}{(2N-1)k}\right)^{\alpha} \leq \left(\frac{1}{h-k}\right)^{\alpha}$, then

$$\varphi(h) \leq [\varphi(k)]^{\beta} c_4^{1-\beta}\left(\frac{1}{k}\right)^{\alpha} \leq [\varphi(k)]^{\beta} c_4^{1-\beta}(2N-1)^{\alpha}\left(\frac{1}{h-k}\right)^{\alpha}.$$

In both cases we have obtained (1.3) with $c_3 = \max\{(2N)^{2\alpha}\tilde{c}_3, c_4^{1-\beta}(2N-1)^{\alpha}\}$.

Remark 1.2. In Lemma 1.2-(III) we assume that there exists $\tilde{k}_0 \geq k_0$ such that $c_7 = N^{\frac{\alpha}{\beta^2}}\varphi(\tilde{k}_0)^{-1} > 1$. We remark that this is always the case because (1.3) implies

$$\varphi(Nh) \leq \frac{c_3}{(h-k_0)^{\alpha}}[\varphi(k_0)]^{\beta} \to 0, \quad h \to +\infty,$$

then there exists $\tilde{k}_0 \geq k_0$ such that

$$\varphi(\tilde{k}_0) < N^{\frac{\alpha}{\beta^2}}.$$

1.2. The second generalization.

In order to give the second generalization of the Stampacchia Lemma, we should mention two papers [16] and [15]. In [16], Mammoliti proved a lemma in order to deal with regularity for solutions to some elliptic equations with degenerate coercivity, see Lemma A.1 in the Appendix in [16]:

Lemma 1.3. *Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-increasing function such that*

$$\varphi(k) \le \frac{c_9}{(h-k)^\alpha} k^{\tilde{\theta}\alpha}[\varphi(k)]^\beta, \quad \forall h > k > 0, \tag{1.21}$$

for some positive constants c_9, with $\alpha > 0$, $0 \le \tilde{\theta} < 1$ and $\beta > 1$. Then there exists $k^* > 0$ such that $\varphi(k^*) = 0$.

Note that, compared with (1.3), a factor $k^{\tilde{\theta}\alpha}$ appears in the right hand side of (1.21).

In [15], Kovalevskii and Voitovich proved three lemmas (Lemmas 2,3,4 in [15]) with condition (1.21) for three cases of β: $0 < \beta < 1$, $\beta = 1$ and $\beta > 1$, see also [17].

Motivated by the above two paper, we give another generalization of Stampacchia Lemma, which can be used in dealing with regularity issues of degenerate elliptic systems, see Section 2.2.

Lemma 1.4. *Let c_{10}, α, β be positive constants, $k_0 > 0$, $N > 1$ and $0 \le \tilde{\theta} < 1$. Let $\varphi : [k_0, +\infty) \to [0, +\infty)$ be nonincreasing and such that*

$$\varphi(Nh) \le \frac{c_{10} k^{\tilde{\theta}\alpha}}{(h-k)^\alpha}[\varphi(k)]^\beta \tag{1.22}$$

for every h, k with $h > k \ge k_0$. It results that:

(I) *if $\beta < 1$ then for any $k \ge k_0$ we have*

$$\varphi(k) \le c_{11} \left(\frac{1}{k}\right)^{\frac{\alpha(1-\tilde{\theta})}{1-\beta}}, \tag{1.23}$$

where

$$c_{11} = \max\left\{ (2N)^{\frac{(1-\tilde{\theta})\alpha(2-\beta)}{(1-\beta)^2}} \left[1 + \varphi(k_0)\left(\frac{(k_0)^{(1-\tilde{\theta})\alpha}}{c_{10}}\right)^{\frac{1}{1-\beta}}\right]^\beta c_{10}^{\frac{1}{1-\beta}}, \varphi(k_0)(2Nk_0)^{\frac{(1-\tilde{\theta})\alpha}{1-\beta}} \right\}.$$

(II) *if $\beta = 1$ then for any $k \ge k_0$ and any $q > 1$, we have*

$$\varphi(k) \le c_{12} \left(\frac{1}{k}\right)^{\tilde{q}}, \tag{1.24}$$

where

$$\tilde{q} = \max\{q, 1 + \alpha\}$$

and c_{12} is the maximum value of

$$(2N)^{\frac{\tilde{q}[\tilde{q}+(1-\tilde{\theta})\alpha]}{(1-\tilde{\theta})\alpha}} \left[1 + \varphi(k_0)\left(\frac{(k_0)^{(1-\tilde{\theta})\alpha}}{c_{10}[1+\varphi(k_0)]^{(1-\tilde{\theta})\alpha/q}}\right)^{\frac{\tilde{q}}{(1-\tilde{\theta})\alpha}}\right]^{1-\frac{(1-\tilde{\theta})\alpha}{\tilde{q}}+\frac{\tilde{q}}{q}} \left(c_{10}[1+\varphi(k_0)]^{(1-\tilde{\theta})\alpha/q}\right)^{\frac{\tilde{q}}{(1-\tilde{\theta})\alpha}}$$

and

$$\varphi(k_0)(2Nk_0)^{\tilde{q}}.$$

(III) *if $\beta > 1$ then we have two cases: there exists $k \ge k_0$ such that $\varphi(k) = 0$, or for any $k \ge k_0$ we have $\varphi(k) > 0$. In such a second case, for any $k \ge \tilde{k}_0$,*

$$\varphi(k) \le \max\left\{1, \varphi(\tilde{k}_0) c_{13}^{N^2(\tilde{k}_0+\tau 2^{1/(1-\tilde{\theta})})\log_N\sqrt{\beta}}\right\} c_{13}^{-k^{\log_N\sqrt{\beta}}},$$

where

$$c_{13} = c_{14}^{\frac{1}{\beta^{\log_N \sqrt{Nc_{15}}}}}, \quad c_{14} = N^{\frac{(1-\tilde{\theta})\alpha}{\beta^2}} \varphi(\tilde{k}_0)^{-1},$$

$$c_{15} = \max\left\{ \max_{1\le \ell \le \ell_0} \frac{\tau^{\frac{1}{1-\tilde{\theta}}} \left(\frac{1}{1-\tilde{\theta}} - 1\right) \cdots \left(\frac{1}{1-\tilde{\theta}} - \ell + 1\right) 2^{\frac{1}{1-\tilde{\theta}} - \ell}}{N(\ln N)^{\ell}}, \frac{\tilde{k}_0 + \tau 2^{\frac{1}{1-\tilde{\theta}}}}{N} \right\},$$

$$\ell_0 = \left[\frac{1}{1-\tilde{\theta}}\right] + 1,$$

$[s]$ is the integer part of s and τ is given in (1.26), and $\tilde{k}_0 \ge k_0$ is a constant such that $c_{14} > 1$.

Proof. (**I**) For the case $\beta < 1$, we take $h = 2k$ in (1.22) and we have

$$\varphi(2Nk) \le \frac{c_{10}}{k^{(1-\tilde{\theta})\alpha}} [\varphi(k)]^{\beta}, \ N > 1, k \ge k_0. \tag{1.25}$$

The difference between (1.6) and (1.25) is that c_3 replaced by c_{10} and α replaced by $(1-\tilde{\theta})\alpha$. Thus, by Lemma 1.2-(I), the result of Lemma 1.4-(I) follows.

(**II**) For the case $\beta = 1$, for any $k \ge k_0$ and any $q > 1$, we let $\tilde{q} = \max\{q, 1+\alpha\}$ and $\hat{q} = \frac{\tilde{q}}{1-\tilde{\theta}}$. (1.22) with $\beta = 1$ implies

$$\begin{aligned} \varphi(Nh) &\le \frac{c_{10} k^{\tilde{\theta}\alpha}}{(h-k)^{\alpha}} \varphi(k) = \frac{c_{10} k^{\tilde{\theta}\alpha}}{(h-k)^{\alpha}} [\varphi(k)]^{\frac{\alpha}{\hat{q}}} [\varphi(k)]^{1-\frac{\alpha}{\hat{q}}} \\ &\le \frac{c_{10} [\varphi(k_0)]^{\frac{\alpha}{\hat{q}}} k^{\tilde{\theta}\alpha}}{(h-k)^{\alpha}} [\varphi(k)]^{1-\frac{\alpha}{\hat{q}}} \\ &\le \frac{c_{10} [1+\varphi(k_0)]^{\frac{(1-\tilde{\theta})\alpha}{q}} k^{\tilde{\theta}\alpha}}{(h-k)^{\alpha}} [\varphi(k)]^{1-\frac{\alpha}{\hat{q}}}, \end{aligned}$$

here we have used the fact

$$\frac{\alpha}{\hat{q}} = \frac{(1-\tilde{\theta})\alpha}{\tilde{q}} \le \frac{(1-\tilde{\theta})\alpha}{q}.$$

The fact $0 < 1 - \frac{\alpha}{\hat{q}} < 1$ allows us to make use of (1.23) to derive (1.24) (with $c_{10}[1+\varphi(k_0)]^{(1-\tilde{\theta})\alpha/q}$ in place of c_{10} and $1 - \frac{\alpha}{\hat{q}}$ in place of β).

(**III**) For the case $\beta > 1$, we let for any nonnegative integer $s \ge 0$,

$$k = N^s \left(\tilde{k}_0 + \tau s^{\frac{1}{1-\tilde{\theta}}}\right) = \tilde{k}_s$$

and

$$Nh = N^{s+1} \left(\tilde{k}_0 + \tau (s+1)^{\frac{1}{1-\tilde{\theta}}}\right) = \tilde{k}_{s+1},$$

where

$$\tau = \max\left\{1, \tilde{k}_0, \left[c_{10}^{1/\alpha} 2^{\tilde{\theta}} (1-\tilde{\theta})\right]^{\frac{1}{1-\tilde{\theta}}}, (c_{10})^{1/\alpha} (\tilde{k}_0)^{\tilde{\theta}}\right\} \tag{1.26}$$

and we recall that $\tilde{k}_0 \ge k_0$ is a constant such that $c_{14} = N^{\frac{(1-\tilde{\theta})\alpha}{\beta^2}} \varphi(\tilde{k}_0)^{-1} > 1$. It is obvious that

$$h = N^s \left(\tilde{k}_0 + \tau (s+1)^{\frac{1}{1-\tilde{\theta}}}\right) > k.$$

We use Taylor's formula in order to get

$$\begin{aligned} h-k &= N^s\tau\left[(s+1)^{\frac{1}{1-\tilde\theta}} - s^{\frac{1}{1-\tilde\theta}}\right] \\ &= N^s\tau\left[\frac{1}{1-\tilde\theta}s^{\frac{\tilde\theta}{1-\tilde\theta}} + \frac{\tilde\theta}{2(1-\tilde\theta)^2}\xi^{\frac{2\tilde\theta-1}{1-\tilde\theta}}\right] \\ &\geq \frac{N^s\tau}{1-\tilde\theta}s^{\frac{\tilde\theta}{1-\tilde\theta}}, \end{aligned} \tag{1.27}$$

where ξ lies in the interval between s and $s+1$. We take h and k as above in (1.22) and we make use of (1.27), (1.26): for any integer $s \geq 1$,

$$\begin{aligned} \varphi(\tilde k_{s+1}) &\leq \frac{c_{10}\tilde k_s^{\tilde\theta\alpha}}{(h-k)^\alpha}[\varphi(\tilde k_s)]^\beta \\ &\leq \frac{c_{10}\left[N^s(\tilde k_0 + \tau s^{\frac{1}{1-\tilde\theta}})\right]^{\tilde\theta\alpha}(1-\tilde\theta)^\alpha}{N^{s\alpha}\tau^\alpha s^{\frac{\alpha\tilde\theta}{1-\tilde\theta}}}[\varphi(\tilde k_s)]^\beta \\ &\leq \frac{c_{10}\left[N^s(2\tau s^{\frac{1}{1-\tilde\theta}})\right]^{\tilde\theta\alpha}(1-\tilde\theta)^\alpha}{N^{s\alpha}\tau^\alpha s^{\frac{\alpha\tilde\theta}{1-\tilde\theta}}}[\varphi(\tilde k_s)]^\beta \\ &\leq \frac{1}{N^{s(1-\tilde\theta)\alpha}}[\varphi(\tilde k_s)]^\beta. \end{aligned} \tag{1.28}$$

From the definition of τ in (1.26) we know $\tau \geq (c_{10})^{1/\alpha}(\tilde k_0)^{\tilde\theta}$, and this implies, by means of (1.22) with $k = \tilde k_0$ and $h = \tilde k_0 + \tau$, that

$$\varphi(\tilde k_1) \leq [\varphi(\tilde k_0)]^\beta. \tag{1.29}$$

Thus, for any positive integer $s \geq 1$,

$$\begin{aligned} &\varphi(\tilde k_{s+1}) \\ \leq\ & \frac{1}{N^{s(1-\tilde\theta)\alpha}}[\varphi(\tilde k_s)]^\beta \\ \leq\ & \frac{1}{N^{s(1-\tilde\theta)\alpha}}\left[\frac{1}{N^{(s-1)(1-\tilde\theta)\alpha}}\left[\varphi(\tilde k_{s-1})\right]^\beta\right]^\beta \\ =\ & \frac{1}{N^{s(1-\tilde\theta)\alpha}}\frac{1}{N^{(s-1)(1-\tilde\theta)\alpha\beta}}\left[\varphi(\tilde k_{s-1})\right]^{\beta^2} \\ \leq\ & \cdots\cdots \\ \leq\ & \frac{[\varphi(\tilde k_1)]^{\beta^s}}{N^{s(1-\tilde\theta)\alpha}N^{(s-1)(1-\tilde\theta)\alpha\beta}N^{(s-2)(1-\tilde\theta)\alpha\beta^2}\cdots N^{1\cdot(1-\tilde\theta)\alpha\beta^{s-1}}} \\ \leq\ & \frac{[\varphi(\tilde k_0)]^{\beta^{s+1}}}{N^{s(1-\tilde\theta)\alpha}N^{(s-1)(1-\tilde\theta)\alpha\beta}N^{(s-2)(1-\tilde\theta)\alpha\beta^2}\cdots N^{1\cdot(1-\tilde\theta)\alpha\beta^{s-1}}} \\ \leq\ & \frac{1}{N^{(1-\tilde\theta)\alpha\beta^{s-1}}}[\varphi(\tilde k_0)]^{\beta^{s+1}} \\ =\ & \left(N^{\frac{(1-\tilde\theta)\alpha}{\beta^2}}\varphi(\tilde k_0)^{-1}\right)^{-\beta^{s+1}}, \end{aligned} \tag{1.30}$$

where we have used (1.28), (1.29) and the fact

$$\begin{aligned} & N > 1, (1-\tilde\theta)\alpha > 0, s \geq 1 \\ \Rightarrow\ & N^{s(1-\tilde\theta)\alpha}N^{(s-1)(1-\tilde\theta)\alpha\beta}\cdots N^{1\cdot(1-\tilde\theta)\alpha\beta^{s-1}} \geq N^{(1-\tilde\theta)\alpha\beta^{s-1}}. \end{aligned}$$

(1.30) yields that, for any positive integer $s \geq 2$,

$$\varphi(\tilde k_s) \leq \left(N^{\frac{(1-\tilde\theta)\alpha}{\beta^2}}\varphi(\tilde k_0)^{-1}\right)^{-\beta^s}. \tag{1.31}$$

For any $k \geq \tilde{k}_0$, we distinguish between two cases: $k \geq \tilde{k}_2$ and $\tilde{k}_0 \leq k < \tilde{k}_2$.

For the case $k \geq \tilde{k}_2$, there exists a positive integer $s \geq 2$ such that

$$\tilde{k}_s \leq k < \tilde{k}_{s+1}.$$

(1.31) allows us to estimate

$$\varphi(k) \leq \varphi(\tilde{k}_s) \leq \left(N^{\frac{(1-\tilde{\theta})\alpha}{\beta^2}} \varphi(\tilde{k}_0)^{-1} \right)^{-\beta^s}. \tag{1.32}$$

Denote

$$\ell_0 = \left[\frac{1}{1-\tilde{\theta}} \right] + 1,$$

where $[t]$ denotes the integer part of t. We let

$$c_{15} = \max \left\{ \max_{1 \leq \ell \leq \ell_0} \frac{\tau \frac{1}{1-\tilde{\theta}} \left(\frac{1}{1-\tilde{\theta}} - 1 \right) \cdots \left(\frac{1}{1-\tilde{\theta}} - \ell + 1 \right) 2^{\frac{1}{1-\tilde{\theta}} - \ell}}{N (\ln N)^\ell}, \frac{\tilde{k}_0 + \tau 2^{\frac{1}{1-\tilde{\theta}}}}{N} \right\},$$

then one can prove that, for $1 \leq s \in \mathbb{R}$,

$$\tilde{k}_0 + \tau (s+1)^{\frac{1}{1-\tilde{\theta}}} \leq c_{15} N^s. \tag{1.33}$$

As a matter of fact, let

$$g(s) = c_{15} N^s - \left[\tilde{k}_0 + \tau (s+1)^{\frac{1}{1-\tilde{\theta}}} \right], \ s \geq 1.$$

It is not difficult to derive that, for any positive integer ℓ with $1 \leq \ell \leq \ell_0$,

$$g^{(\ell)}(s) = c_{15} N^s (\ln N)^\ell - \tau \frac{1}{1-\tilde{\theta}} \left(\frac{1}{1-\tilde{\theta}} - 1 \right) \cdots \left(\frac{1}{1-\tilde{\theta}} - \ell + 1 \right) (s+1)^{\frac{1}{1-\tilde{\theta}} - \ell}$$

and

$$g^{(\ell)}(1) = c_{15} N (\ln N)^\ell - \tau \frac{1}{1-\tilde{\theta}} \left(\frac{1}{1-\tilde{\theta}} - 1 \right) \cdots \left(\frac{1}{1-\tilde{\theta}} - \ell + 1 \right) 2^{\frac{1}{1-\tilde{\theta}} - \ell}.$$

We assert that, the $(\ell_0 + 1)$-th derivative of $g(s)$:

$$g^{(\ell_0+1)}(s) = c_{15} N^s (\ln N)^{\ell_0+1} - \tau \frac{1}{1-\tilde{\theta}} \left(\frac{1}{1-\tilde{\theta}} - 1 \right) \cdots \left(\frac{1}{1-\tilde{\theta}} - \ell_0 \right) (s+1)^{\frac{1}{1-\tilde{\theta}} - \ell_0 - 1}$$

is positive, because

$$\tau \frac{1}{1-\tilde{\theta}} \left(\frac{1}{1-\tilde{\theta}} - 1 \right) \cdots \left(\frac{1}{1-\tilde{\theta}} - \ell_0 \right) \leq 0.$$

$g^{(\ell_0+1)}(s) > 0$ implies $g^{(\ell_0)}(s)$ is an increasing function, which together with $g^{(\ell_0)}(1) \geq 0$ (which is a direct consequence by the definition of c_{15}) implies $g^{(\ell_0)}(s)$ is positive, which in turn, implies $g^{(\ell_0-1)}(x)$ is an increasing function. After ℓ_0 times, we derive that $g(s)$ is positive, thus (1.33) is proved.

From (1.33) and the fact $k < \tilde{k}_{s+1}$ we know that

$$k < N^{s+1} \left(\tilde{k}_0 + \tau (s+1)^{\frac{1}{1-\tilde{\theta}}} \right) \leq c_{15} N^{2s+1},$$

which yields

$$\begin{aligned} s &> \frac{1}{2} \left[\log_N \left(\frac{k}{c_{15}} \right) - 1 \right] \\ &= \frac{1}{2} \left[\log_N k - \log_N c_{15} - 1 \right] \\ &= \log_N \beta \log_\beta k^{\frac{1}{2}} - \frac{1}{2} \log_N (N c_{15}) \\ &= \log_\beta k^{\frac{1}{2} \log_N \beta} - \log_N \sqrt{N c_{15}} \\ &= \log_\beta k^{\log_N \sqrt{\beta}} - \log_N \sqrt{N c_{15}}. \end{aligned} \tag{1.34}$$

Combining (1.32) with (1.34) we arrive at

$$\begin{aligned}\varphi(k) &\leq \left(N^{\frac{(1-\tilde\theta)\alpha}{\beta^2}}\varphi(\tilde k_0)^{-1}\right)^{-\beta^s} = c_{14}^{-\beta^s} \leq c_{14}^{-\beta^{\left[\log_\beta k^{\log_N \sqrt{\beta}} - \log_N \sqrt{Nc_{15}}\right]}} \\ &= c_{14}^{-\frac{\beta^{\log_\beta k^{\log_N \sqrt{\beta}}}}{\beta^{\log_N \sqrt{Nc_{15}}}}} = c_{13}^{-\beta^{\log_\beta k^{\log_N \sqrt{\beta}}}} = c_{13}^{-k^{\log_N \sqrt{\beta}}},\end{aligned} \tag{1.35}$$

where $c_{13} = c_{14}^{\frac{1}{\beta^{\log_N \sqrt{Nc_{15}}}}}$.

For the case $\tilde k_0 \leq k < \tilde k_2$ we have

$$\begin{aligned}&\varphi(k) \leq \varphi(\tilde k_0) = \varphi(\tilde k_0) c_{13}^{\tilde k_2 \log_N \sqrt{\beta}} c_{13}^{-\tilde k_2 \log_N \sqrt{\beta}} \\ &\leq \varphi(\tilde k_0) c_{13}^{\tilde k_2 \log_N \sqrt{\beta}} c_{13}^{-k \log_N \sqrt{\beta}}.\end{aligned} \tag{1.36}$$

(1.35) and (1.36) yield the desired result. □

2. Applications

This section is devoted to give some applications of the generalized Stampacchia Lemmas proved in the first section to regularity theory of quasilinear elliptic systems.

2.1. Quasilinear elliptic systems with ellipticity condition.

Let $n > 2$, $N \geq 2$ be integers and Ω an open bounded subset of $\mathbb{R}^n$. We consider (distributional) solutions of quasilinear elliptic systems involving N equations of the form

$$\begin{cases} -\displaystyle\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\sum_{\beta=1}^{N}\sum_{j=1}^{n} a_{i,j}^{\alpha,\beta}(x,u(x))\frac{\partial u^\beta(x)}{\partial x_j}\right) = f^\alpha, & \text{in } \Omega, \\ u(x) = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\alpha \in \{1,2,\ldots,N\}$ is the equation index.

We make use of the following assumptions on the coefficients $a_{i,j}^{\alpha,\beta}(x,y)$: for $i,j \in \{1,\ldots,n\}$ and $\alpha,\beta \in \{1,\ldots,N\}$,

$(\mathcal{A}_1)$ (Carathéodory condition) $x \mapsto a_{i,j}^{\alpha,\beta}(x,y)$ is measurable and $y \mapsto a_{i,j}^{\alpha,\beta}(x,y)$ is continuous;

$(\mathcal{A}_2)$ (boundedness of all the coefficients) there exists a positive constant c such that

$$|a_{i,j}^{\alpha,\beta}(x,y)| \leq c$$

for almost all $x \in \Omega$ and all $y \in \mathbb{R}^N$;

$(\mathcal{A}_3)$ (ellipticity of the diagonal coefficients) there exists a positive constant c_0 such that

$$c_0|\lambda|^2 \leq \sum_{i,j=1}^{n} a_{i,j}^{\alpha,\alpha}(x,y)\lambda_i\lambda_j$$

for almost all $x \in \Omega$, all $y \in \mathbb{R}^N$, all $\lambda \in \mathbb{R}^n$ and all $\alpha \in \{1,\ldots,N\}$;

$(\mathcal{A}_4)$ (proportional condition of the off-diagonal coefficients) there exist constants $r^{\alpha,\beta}$, $\alpha,\beta \in \{1,\ldots,N\}$, such that for all $i,j \in \{1,\ldots,n\}$, almost all $x \in \Omega$ and all $y \in \mathbb{R}^N$,

$$a_{i,j}^{\alpha,\beta}(x,y) = r^{\alpha,\beta} a_{i,j}^{\beta,\beta}(x,y),$$

the constants $r^{\alpha,\beta}$, $\alpha,\beta \in \{1,\ldots,N\}$, be such that $r^{\alpha,\alpha}=1$ and

$$\det \mathcal{R} = \det \begin{pmatrix} 1 & r^{2,1} & r^{3,1} & \cdots & r^{N,1} \\ r^{1,2} & 1 & r^{3,2} & \cdots & r^{N,2} \\ r^{1,3} & r^{2,3} & 1 & \cdots & r^{N,3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r^{1,N} & r^{2,N} & r^{3,N} & \cdots & 1 \end{pmatrix} \neq 0.$$

The following example gives $a_{i,j}^{\alpha,\beta}(x,y): \Omega \times \mathbb{R}^N \to \mathbb{R}$ with $\alpha,\beta \in \{1,\ldots,N\}$ and $i,j \in \{1,\ldots,n\}$, which satisfy the conditions $(\mathcal{A}_1)$-$(\mathcal{A}_4)$.

Example 2.1. We let $\delta_{i,j}$ be the Kronecker symbol. For $\alpha,\beta \in \{1,\ldots,N\}$ and $i,j \in \{1,\ldots,n\}$, we define $a_{i,j}^{\alpha,\beta}(x,y)$ as follows: for $\alpha \in \{1,\ldots,N\}$,

$$a_{i,j}^{\alpha,\alpha}(x,y) = \delta_{i,j},$$

and for $\alpha,\beta \in \{1,\ldots,N\}$ with $\alpha \neq \beta$,

$$a_{i,j}^{\alpha,\beta}(x,y) = r^{\alpha,\beta} a_{i,j}^{\beta,\beta}(x,y) = r^{\alpha,\beta}\delta_{i,j},$$

where the real numbers $r^{\alpha,\beta}$ be such that $\det \mathcal{R} \neq 0$, thus the condition $(\mathcal{A}_4)$ holds true naturally. Moreover, the condition $(\mathcal{A}_1)$ is satisfied because $x \mapsto a_{i,j}^{\alpha,\beta}(x,y)$ is measurable and $y \mapsto a_{i,j}^{\alpha,\beta}(x,y)$ is continuous; the condition $(\mathcal{A}_2)$ is satisfied with $c = \|\mathcal{R}\| = \left(\sum_{\alpha,\beta=1}^N |r^{\alpha,\beta}|^2\right)^{1/2}$; the condition $(\mathcal{A}_3)$ is satisfied with $c_0 = 1$.

Definition 2.1. We say that a function $u: \Omega \to \mathbb{R}^N$ is a (distributional) solution with respect to (2.1), if $u \in W_0^{1,2}(\Omega;\mathbb{R}^N)$ and

$$\int_\Omega \sum_{\alpha,\beta=1}^N \sum_{i,j=1}^n a_{i,j}^{\alpha,\beta}(x,u(x)) D_j u^\beta(x) D_i \varphi^\alpha(x) dx = \sum_{\alpha=1}^N \int_\Omega f^\alpha(x)\varphi^\alpha(x)dx \tag{2.2}$$

holds true for all $\varphi \in W_0^{1,2}(\Omega;\mathbb{R}^N)$.

We remark that the condition $(\mathcal{A}_2)$ is added in order to make finite the integral on the left hand side of (2.2). We remark also that assumption $(\mathcal{A}_4)$ has been made in the special case $N=2$ in [18]. Note that we do not need that the off-diagonal coefficients are small, compare with the assumption $(\mathcal{A}_3)$ on page 213 in [18].

For the case $N = 1$, that is, (2.1) is only one single equation, existence of distributional solutions $u: \Omega \subset \mathbb{R}^n \to \mathbb{R}$ has been deeply studied, starting from [19], see also [5,20–23] and the survey [24]. Regularity results are contained in [25–30] and the survey [31] (see also [32]). For systems, $N \geq 2$, the situation is very different with respect to the single equation case. There is a gap in the regularity scale for the solutions of systems and for the minimizers of integral vectorial functionals. Existence and regularity of solutions for one single equation is usually obtained by a truncation argument, which shows why the vectorial case $N \geq 2$ is difficult and only few contributions are available in the literature. In fact, for systems $N \geq 2$, the p-Laplacian $\mathcal{A}(x,y,\xi) = |\xi|^{p-2}\xi$ is treated in [33,34], and the anisotropic case, in which each component of the gradient $D_i u$ may have a possibly different exponent p_i, is dealt in [35,36]. For some other results related to elliptic systems, we refer to [37–39].

Recently, some noteworthy developments are made. In [18], the authors studied the existence of solutions of quasilinear elliptic systems involving N equations and a measure on the right hand side with the form

$$\begin{cases} -\displaystyle\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\sum_{\beta=1}^N \sum_{j=1}^n a_{i,j}^{\alpha,\beta}(x,u(x))\frac{\partial}{\partial x_j}u^\beta\right) = \mu^\alpha, & \text{in } \Omega, \\ u(x) = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\alpha \in \{1, 2, \ldots, N\}$ is the equation index and μ is a finite Randon measure on $\mathbb{R}^n$ with values in $\mathbb{R}^N$. Existence of a solution was proved for two different sets of assumptions on the coefficients $a_{i,j}^{\alpha,\beta}(x,y)$. In [40], the authors obtained similar results by assuming some smallness and cancellation conditions on the coefficients. Maximum principles for some quasilinear elliptic systems can be found in [41].

In this section, we consider elliptic system (2.1) under the assumptions ($\mathcal{A}_1$)-($\mathcal{A}_4$). We will use the generalized Stampacchia Lemmas proved in Section 1 to derive some global regularity results for distributional solutions to (2.1). To our knowledge, it seems to be the first applications of the powerful and elegant Stampacchia Lemma to elliptic systems.

Theorem 2.1. *Suppose u is a solution with respect to $f \in L^m(\Omega; \mathbb{R}^N)$, $m > (2^*)' = \frac{2n}{n+2}$. Under the assumptions ($\mathcal{A}_1$)–($\mathcal{A}_4$), we have*

(1) *if $\frac{2n}{n+2} < m < \frac{n}{2}$, then $|u| \in L^{m^{**}}_{weak}(\Omega)$, where $m^{**} = (m^*)^* = \frac{nm}{n-2m}$;*
(2) *if $m = \frac{n}{2}$, then for any $q > 1$, $|u| \in L^{\tilde{q}}_{weak}(\Omega)$, $\tilde{q} = \max\{q, 1 + 2^*\}$;*
(3) *if $m > \frac{n}{2}$, then there exists $\lambda > 0$ such that $e^{\lambda |u|^{\log_N \sqrt{\beta}}} \in L^1(\Omega)$.*

In the statement of Theorem 2.1-(1), the weak L^q spaces $L^q_{weak}(\Omega)$, known also as Marcinkiewicz spaces, are defined as follows: if $q > 1$, then the space $L^q_{weak}(\Omega)$ consists of measurable functions g on Ω such that

$$\sup_{t>0} t |\{x \in \Omega : |g(x)| > t\}|^{\frac{1}{q}} < \infty. \tag{2.3}$$

For a detailed analysis of $L^q_{weak}(\Omega)$ spaces we refer to [42].

Proof. (of Theorem 2.1). We take a test function $\varphi = (\varphi^1, \ldots, \varphi^N)$ in (2.2) as

$$\varphi^\alpha = \sum_{\gamma=1}^{N} C_\alpha^\gamma G_k(u^\gamma), \ \alpha \in \{1, \ldots, N\}, \tag{2.4}$$

here and in what follows, for $s \in \mathbb{R}$,

$$G_k(s) = s - T_k(s) = s - \min\left\{1, \frac{k}{|s|}\right\} s,$$

and C_α^γ, $\alpha, \gamma \in \{1, \ldots, N\}$, are real constants to be chosen later. Then

$$D_i \varphi^\alpha = \sum_{\gamma=1}^{N} C_\alpha^\gamma D_i u^\gamma 1_{A_k^\gamma},$$

where

$$A_k^\gamma = \{x \in \Omega : |u^\gamma| > k\}, \quad \gamma = 1, \ldots, N,$$

and $1_E(x)$ is the characteristic function of the set E, that is, $1_E(x) = 1$ if $x \in E$ and $1_E(x) = 0$ if $x \notin E$. (2.2) with the test function φ be as in (2.4) gives

$$\int_\Omega \sum_{\alpha,\beta=1}^{N} \sum_{i,j=1}^{n} a_{i,j}^{\alpha,\beta} D_j u^\beta \sum_{\gamma=1}^{N} C_\alpha^\gamma D_i u^\gamma 1_{A_k^\gamma} dx = \sum_{\alpha=1}^{N} \int_\Omega f^\alpha \sum_{\gamma=1}^{N} C_\alpha^\gamma G_k(u^\gamma) dx, \tag{2.5}$$

where $a_{i,j}^{\alpha,\beta} = a_{i,j}^{\alpha,\beta}(x,u)$.

For the left hand side of (2.5), we note that

$$
\begin{aligned}
&\int_\Omega \sum_{\alpha,\beta=1}^{N}\sum_{i,j=1}^{n} a_{i,j}^{\alpha,\beta} D_j u^\beta \sum_{\gamma=1}^{N} C_\alpha^\gamma D_i u^\gamma 1_{A_k^\gamma} dx \\
= &\int_\Omega \sum_{\alpha=1}^{N}\sum_{i,j=1}^{n} a_{i,j}^{\alpha,\alpha} D_j u^\alpha \sum_{\gamma=1}^{N} C_\alpha^\gamma D_i u^\gamma 1_{A_k^\gamma} dx \quad (\text{terms for } \beta=\alpha) \\
&+\int_\Omega \sum_{\alpha=1}^{N}\sum_{\beta=1,\beta\neq\alpha}^{N}\sum_{i,j=1}^{n} a_{i,j}^{\alpha,\beta} D_j u^\beta \sum_{\gamma=1}^{N} C_\alpha^\gamma D_i u^\gamma 1_{A_k^\gamma} dx \quad (\text{terms for } \beta\neq\alpha) \\
= &\int_\Omega \sum_{\alpha=1}^{N}\sum_{i,j=1}^{n} C_\alpha^\alpha a_{i,j}^{\alpha,\alpha} D_j u^\alpha D_i u^\alpha 1_{A_k^\alpha} dx \quad (\text{terms for } \gamma=\beta=\alpha) \\
&+\int_\Omega \sum_{\alpha=1}^{N}\sum_{i,j=1}^{n} a_{i,j}^{\alpha,\alpha} D_j u^\alpha \sum_{\gamma=1,\gamma\neq\alpha}^{N} C_\alpha^\gamma D_i u^\gamma 1_{A_k^\gamma} dx \quad (\text{terms for } \gamma\neq\beta=\alpha) \\
&+\int_\Omega \sum_{\alpha=1}^{N}\sum_{\beta=1,\beta\neq\alpha}^{N}\sum_{i,j=1}^{n} a_{i,j}^{\alpha,\beta} D_j u^\beta C_\alpha^\beta D_i u^\beta 1_{A_k^\beta} dx \quad (\text{terms for } \gamma=\beta\neq\alpha) \\
&+\int_\Omega \sum_{\alpha=1}^{N}\sum_{\beta=1,\beta\neq\alpha}^{N}\sum_{i,j=1}^{n} a_{i,j}^{\alpha,\beta} D_j u^\beta \sum_{\gamma=1,\gamma\neq\beta}^{N} C_\alpha^\gamma D_i u^\gamma 1_{A_k^\gamma} dx \quad (\text{terms for } \gamma\neq\beta, \beta\neq\alpha) \\
= &\int_\Omega \sum_{\alpha=1}^{N}\sum_{i,j=1}^{n} C_\alpha^\alpha a_{i,j}^{\alpha,\alpha} D_j u^\alpha D_i u^\alpha 1_{A_k^\alpha} dx \quad (\text{terms for } \gamma=\beta=\alpha) \\
&+\int_\Omega \sum_{\alpha=1}^{N}\sum_{i,j=1}^{n} a_{i,j}^{\alpha,\alpha} D_j u^\alpha \sum_{\gamma=1,\gamma\neq\alpha}^{N} C_\alpha^\gamma D_i u^\gamma 1_{A_k^\gamma} dx \quad (\text{terms for } \gamma\neq\beta=\alpha) \\
&+\int_\Omega \sum_{\alpha=1}^{N}\sum_{\beta=1,\beta\neq\alpha}^{N}\sum_{i,j=1}^{n} r^{\alpha,\beta} a_{i,j}^{\beta,\beta} D_j u^\beta C_\alpha^\beta D_i u^\beta 1_{A_k^\beta} dx \quad (\text{terms for } \gamma=\beta\neq\alpha) \\
&+\int_\Omega \sum_{\alpha=1}^{N}\sum_{\beta=1,\beta\neq\alpha}^{N}\sum_{i,j=1}^{n} r^{\alpha,\beta} a_{i,j}^{\beta,\beta} D_j u^\beta \sum_{\gamma=1,\gamma\neq\beta}^{N} C_\alpha^\gamma D_i u^\gamma 1_{A_k^\gamma} dx \quad (\text{terms for } \gamma\neq\beta, \beta\neq\alpha) \\
= &\; I_1+I_2+I_3+I_4.
\end{aligned}
\tag{2.6}
$$

It is obvious that, recalling that $r^{\alpha,\alpha}=1$ for all $\alpha\in\{1,\dots,N\}$,

$$
\begin{aligned}
I_1+I_3 &= \int_\Omega \sum_{\alpha,\beta=1}^{N}\sum_{i,j=1}^{n} r^{\alpha,\beta} a_{i,j}^{\beta,\beta} D_j u^\beta C_\alpha^\beta D_i u^\beta 1_{A_k^\beta} dx \\
&= \int_\Omega \sum_{\beta=1}^{N}\sum_{i,j=1}^{n} \left(\sum_{\alpha=1}^{N} r^{\alpha,\beta} C_\alpha^\beta\right) a_{i,j}^{\beta,\beta} D_j u^\beta D_i u^\beta 1_{A_k^\beta} dx
\end{aligned}
\tag{2.7}
$$

and

$$
\begin{aligned}
I_2+I_4 &= \int_\Omega \sum_{\alpha,\beta=1}^{N}\sum_{i,j=1}^{n} r^{\alpha,\beta} a_{i,j}^{\beta,\beta} D_j u^\beta \sum_{\gamma=1,\gamma\neq\beta}^{N} C_\alpha^\gamma D_i u^\gamma 1_{A_k^\gamma} dx \\
&= \int_{A_k^\gamma} \sum_{\beta,\gamma=1,\beta\neq\gamma}^{N}\sum_{i,j=1}^{n} \left(\sum_{\alpha=1}^{N} r^{\alpha,\beta} C_\alpha^\gamma\right) a_{i,j}^{\beta,\beta} D_j u^\beta D_i u^\gamma dx.
\end{aligned}
\tag{2.8}
$$

If one can choose

$$
\sum_{\alpha=1}^{N} r^{\alpha,\beta} C_\alpha^\beta = 1, \quad \text{for } \beta\in\{1,\dots,N\} \tag{2.9}
$$

and

$$\sum_{\alpha=1}^{N} r^{\alpha,\beta} C_\alpha^\gamma = 0, \quad \text{for } \beta,\gamma \in \{1,\dots,N\}, \beta \neq \gamma, \tag{2.10}$$

then (2.9) and assumption $(\mathcal{A}_3)$ allow us to estimate

$$I_1 + I_3 = \int_\Omega \sum_{\beta=1}^{N} \sum_{i,j=1}^{n} a_{i,j}^{\beta,\beta} D_j u^\beta D_i u^\beta 1_{A_k^\beta} dx \geq c_0 \sum_{\beta=1}^{N} \int_{A_k^\beta} |Du^\beta|^2 dx, \tag{2.11}$$

and (2.8) together with (2.10) implies

$$I_2 + I_4 = 0. \tag{2.12}$$

Combining (2.5), (2.6), (2.11) and (2.12) one has

$$c_0 \sum_{\beta=1}^{N} \int_{A_k^\beta} |Du^\beta|^2 dx \leq \sum_{\alpha=1}^{N} \int_\Omega f^\alpha \sum_{\gamma=1}^{N} C_\alpha^\gamma G_k(u^\gamma) dx. \tag{2.13}$$

Now, we prove that Eqs. (2.9) and (2.10) are valid for appropriate choice of the constants C_α^γ, $\alpha,\gamma \in \{1,\dots,N\}$. In fact, (2.9) and (2.10) have the form

$$\sum_{\alpha=1}^{N} r^{\alpha,\beta} C_\alpha^\gamma = \delta_{\beta\gamma}, \quad \text{for } \beta,\gamma \in \{1,\dots,N\},$$

where $\delta_{\beta\gamma}$ is the Kronecker symbol. We note that the above system has N^2 equations and N^2 unknowns C_α^γ, $\alpha,\gamma \in \{1,\dots,N\}$, and can be rewritten as

$$\begin{pmatrix} \mathcal{R} & 0 & \cdots & 0 \\ 0 & \mathcal{R} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{R} \end{pmatrix} \begin{pmatrix} C^1 \\ C^2 \\ \vdots \\ C^N \end{pmatrix} = \begin{pmatrix} e^1 \\ e^2 \\ \vdots \\ e^N \end{pmatrix}, \tag{2.14}$$

where

$$\mathcal{R} = \begin{pmatrix} 1 & r^{2,1} & r^{3,1} & \cdots & r^{N,1} \\ r^{1,2} & 1 & r^{3,2} & \cdots & r^{N,2} \\ r^{1,3} & r^{2,3} & 1 & \cdots & r^{N,3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r^{1,N} & r^{2,N} & r^{3,N} & \cdots & 1 \end{pmatrix}, \quad C^j = \begin{pmatrix} C_1^j \\ C_2^j \\ C_3^j \\ \vdots \\ C_N^j \end{pmatrix}$$

and e^j is the unit vector of $\mathbb{R}^N$, $j \in \{1,\dots,N\}$. By assumption $(\mathcal{A}_4)$, $\det \mathcal{R} \neq 0$, and noticing the right hand side of system (2.14) is nonzero, then there exists a unique nonzero solution to (2.14). We choose C_α^γ to be the unique nonzero solution to (2.14) and we have (2.9) and (2.10).

We now estimate the right hand side of (2.13). Note that

$$G_k(u^\gamma) = 0, \quad \text{for } x \in \Omega \setminus A_k^\gamma,$$

then Sobolev Embedding Theorem

$$\|v\|_{L^{p^*}(\Omega)} \leq c_* \|Dv\|_{L^p(\Omega)}, \quad \forall v \in W_0^{1,p}(\Omega),$$

Hölder inequality and Young inequality yield

$$\begin{aligned}
&\sum_{\alpha=1}^{N}\int_{\Omega} f^{\alpha}\sum_{\gamma=1}^{N} C_{\alpha}^{\gamma} G_k(u^{\gamma})dx \\
=\ &\sum_{\gamma=1}^{N}\int_{A_k^{\gamma}}\left(\sum_{\alpha=1}^{N} C_{\alpha}^{\gamma} f^{\alpha}\right) G_k(u^{\gamma})dx \\
\leq\ &\sum_{\gamma=1}^{N}\left(\int_{A_k^{\gamma}}\left|\sum_{\alpha=1}^{N} C_{\alpha}^{\gamma} f^{\alpha}\right|^{(2^*)'} dx\right)^{\frac{1}{(2^*)'}}\left(\int_{\Omega}|G_k(u^{\gamma})|^{2^*}dx\right)^{\frac{1}{2^*}} \\
\leq\ &c_*\sum_{\gamma=1}^{N}\left(\int_{A_k^{\gamma}}\left|\sum_{\alpha=1}^{N} C_{\alpha}^{\gamma} f^{\alpha}\right|^{(2^*)'} dx\right)^{\frac{1}{(2^*)'}}\left(\int_{\Omega}|DG_k(u^{\gamma})|^{2}dx\right)^{\frac{1}{2}} \\
\leq\ &c_*^2 C(\varepsilon)\sum_{\gamma=1}^{N}\left(\int_{A_k^{\gamma}}\left|\sum_{\alpha=1}^{N} C_{\alpha}^{\gamma} f^{\alpha}\right|^{(2^*)'} dx\right)^{\frac{2}{(2^*)'}}+\varepsilon\sum_{\gamma=1}^{N}\int_{A_k^{\gamma}}|Du^{\gamma}|^2dx.
\end{aligned} \tag{2.15}$$

Substituting (2.15) into (2.13), and taking $\varepsilon=\frac{c_0}{2}$, we arrive at

$$\begin{aligned}
&\sum_{\beta=1}^{N}\int_{A_k^{\beta}}|Du^{\beta}|^2dx \leq c_{16}\sum_{\beta=1}^{N}\left(\int_{A_k^{\beta}}\left|\sum_{\alpha=1}^{N} C_{\alpha}^{\beta} f^{\alpha}\right|^{(2^*)'} dx\right)^{\frac{2}{(2^*)'}} \\
\leq\ &c_{16}\sum_{\beta=1}^{N}\left(\int_{A_k^{\beta}}\left|\sum_{\alpha=1}^{N} C_{\alpha}^{\beta} f^{\alpha}\right|^{m} dx\right)^{\frac{2}{m}}|A_k^{\beta}|^{\left(1-\frac{(2^*)'}{m}\right)\frac{2}{(2^*)'}} \\
\leq\ &c_{17}\sum_{\beta=1}^{N}|A_k^{\beta}|^{\left(1-\frac{(2^*)'}{m}\right)\frac{2}{(2^*)'}} \leq c_{17}N|A_k|^{\left(1-\frac{(2^*)'}{m}\right)\frac{2}{(2^*)'}},
\end{aligned} \tag{2.16}$$

where

$$c_{17}=c_{16}\sum_{\beta=1}^{N}\left(\int_{\Omega}\left|\sum_{\alpha=1}^{N} C_{\alpha}^{\beta} f^{\alpha}\right|^{m} dx\right)^{\frac{2}{m}}$$

is a constant and

$$A_k=\{x\in\Omega:|u(x)|>k\}.$$

The left hand side of (2.16) can be estimated as: for any $h>k\geq k_0$,

$$\begin{aligned}
&\sum_{\beta=1}^{N}\int_{A_k^{\beta}}|Du^{\beta}|^2dx \\
=\ &\sum_{\beta=1}^{N}\int_{\Omega}|DG_k(u^{\beta})|^2dx \\
\geq\ &\frac{1}{c_*^2}\sum_{\beta=1}^{N}\left(\int_{A_k^{\beta}}|G_k(u^{\beta})|^{2^*}dx\right)^{\frac{2}{2^*}} \\
\geq\ &\frac{1}{c_*^2}\sum_{\beta=1}^{N}\left(\int_{A_h^{\beta}}|G_k(u^{\beta})|^{2^*}dx\right)^{\frac{2}{2^*}} \\
\geq\ &\frac{1}{c_*^2}(h-k)^2\sum_{\beta=1}^{N}|A_h^{\beta}|^{\frac{2}{2^*}}.
\end{aligned} \tag{2.17}$$

Combining (2.16) and (2.17) we have

$$\frac{1}{c_*^2}(h-k)^2\sum_{\beta=1}^{N}|A_h^\beta|^{\frac{2}{2^*}} \le c_{17}N|A_k|^{\left(1-\frac{(2^*)'}{m}\right)\frac{2}{(2^*)'}},$$

which implies

$$\sum_{\beta=1}^{N}|A_h^\beta| \le \frac{c_{18}}{(h-k)^{2^*}}|A_k|^{\left(1-\frac{(2^*)'}{m}\right)(2^*-1)}, \tag{2.18}$$

where $c_{18} = N(c_{17}c_*^2N)^{2^*/2}$. It is not difficult to prove that

$$A_{Nh} \subset \bigcup_{\beta=1}^{N} A_h^\beta. \tag{2.19}$$

In fact, for any $x \in A_{Nh}$, one has $|u(x)| > Nh$, this fact together with

$$|u(x)| \le |u^1(x)| + \cdots + |u^N(x)|$$

implies that there exists at least one $\beta \in \{1,\ldots,N\}$ such that $|u^\beta(x)| > h$, thus $x \in \bigcup_{\beta=1}^N A_h^\beta$ and (2.19) holds true. (2.19) implies

$$|A_{Nh}| \le \sum_{\beta=1}^{N}|A_h^\beta|, \tag{2.20}$$

which together with (2.18) yields

$$|A_{Nh}| \le \frac{c_{18}}{(h-k)^{2^*}}|A_k|^{\left(1-\frac{(2^*)'}{m}\right)(2^*-1)}. \tag{2.21}$$

The condition (1.3) in Lemma 1.2 holds true with

$$k_0 = 0, \varphi(k) = |A_k|, c_3 = c_{18}, \alpha = 2^* \text{ and } \beta = \left(1-\frac{(2^*)'}{m}\right)(2^*-1).$$

Now we use Lemma 1.2 to derive that:

(1) if $0 < \beta < 1$, that is, $\frac{2n}{n+2} < m < \frac{n}{2}$, then

$$|A_k| \le c_4\left(\frac{1}{k}\right)^{\frac{\alpha}{1-\beta}} = c_4\left(\frac{1}{k}\right)^{\frac{nm}{n-2m}}, \tag{2.22}$$

which is equivalent to $|u| \in L^{m^{**}}_{weak}(\Omega)$.

(2) if $\beta = 1$, that is, $m = \frac{n}{2}$, then for any $q > 1$,

$$|A_k| \le c_5\left(\frac{1}{k}\right)^{\tilde q}, \quad \tilde q = \max\{q, 1+2^*\},$$

which is equivalent to

$$|u| \in L^{\tilde q}_{weak}(\Omega).$$

(3) if $\beta > 1$, that is, $m > \frac{n}{2}$, then we let $\tilde k_0 > k_0 = 0$ satisfy $c_7 = N^{\frac{\alpha}{\beta^2}}\varphi(\tilde k_0)^{-1} > 1$. Lemma 1.2-(III) allows us to derive that, for every $k \ge \tilde k_0$,

$$|\{|u| > k\}| \le c_{19} \cdot c_6^{-k^{\log_N \sqrt{\beta}}}. \tag{2.23}$$

(2.23) implies

$$\left|\left\{c_6^{\frac{1}{2}|u|^{\log_N \sqrt{\beta}}} > c_6^{\frac{1}{2}k^{\log_N \sqrt{\beta}}}\right\}\right| = |\{|u| > k\}| \le c_{19} \cdot c_6^{-k^{\log_N \sqrt{\beta}}}.$$

Let $\tilde{k} = c_6^{\frac{1}{2}k^{\log_N \sqrt{\beta}}}$, then

$$\left|\left\{c_6^{\frac{1}{2}|u|^{\log_N \sqrt{\beta}}} > \tilde{k}\right\}\right| \le \frac{c_{19}}{\tilde{k}^2}.$$

We now use Lemma 3.11 in [3] which states that the sufficient and necessary condition for a function $g \in L^r(\Omega)$, $r \ge 1$, is

$$\sum_{k=0}^{\infty} k^{r-1}|\{|g| > k\}| < +\infty.$$

We notice that since $k \ge \tilde{k}_0$, then $\tilde{k} = c_6^{\frac{1}{2}k^{\log_N \sqrt{\beta}}} \ge c_6^{\frac{1}{2}\tilde{k}_0^{\log_N \sqrt{\beta}}}$. We use the above lemma for $g = c_6^{\frac{1}{2}|u|^{\log_N \sqrt{\beta}}}$ and $r = 1$, since

$$\begin{aligned} & \sum_{k=0}^{\infty}\left|\left\{c_6^{\frac{1}{2}|u|^{\log_N \sqrt{\beta}}} > \tilde{k}\right\}\right| \\ = & \left(\sum_{k=0}^{K} + \sum_{k=K+1}^{+\infty}\right)\left|\left\{c_6^{\frac{1}{2}|u|^{\log_N \sqrt{\beta}}} > \tilde{k}\right\}\right| \\ \le & (K+1)|\Omega| + c_{19}\sum_{k=K+1}^{\infty}\frac{1}{\tilde{k}^2} \\ < & +\infty, \end{aligned}$$

where $K = \left[c_6^{\frac{1}{2}\tilde{k}_0^{\log_N \sqrt{\beta}}}\right]$, then

$$c_6^{\frac{1}{2}|u|^{\log_N \sqrt{\beta}}} \in L^1(\Omega). \tag{2.24}$$

We let $\lambda = \frac{1}{2}\ln c_6$, then $e^{\lambda} = c_6^{\frac{1}{2}}$, which together with (2.24) implies the desired result $e^{\lambda|u|^{\log_N \sqrt{\beta}}} \in L^1(\Omega)$. □

Remark 2.1. In Theorem 2.1-(1) we derive the result $|u| \in L^{m^{**}}_{weak}(\Omega)$, this implies, for any $j \in \{1, \dots, N\}$,

$$u^j(x) \in L^{m^{**}}_{weak}(\Omega). \tag{2.25}$$

In fact, for any $j \in \{1, \dots, N\}$,

$$|u^j(x)| \le |u(x)|,$$

which implies, for any $k > 0$,

$$\{x \in \Omega : |u^j(x)| > k\} \subset \{x \in \Omega : |u(x)| > k\},$$

thus $|A_k^j| \le |A_k|$, this together with (2.22) implies

$$|A_k^j| \le c_4\left(\frac{1}{k}\right)^{\frac{nm}{n-2m}} = c_4\left(\frac{1}{k}\right)^{m^{**}},$$

(2.25) follows.

2.2. Quasilinear elliptic systems with degenerate ellipticity condition.

In this subsection we consider (distributional) solutions of quasilinear elliptic systems (2.1). We make use of the following assumptions on the coefficients $a_{i,j}^{\alpha,\beta}(x,y)$: for $i,j \in \{1,\dots,n\}$ and $\alpha,\beta \in \{1,\dots,N\}$, $(\mathcal{A}_1)$, $(\mathcal{A}_2)$ and $(\mathcal{A}_4)$ hold, but $(\mathcal{A}_3)$ replaced by

$(\mathcal{A}_3^*)$ (degenerate ellipticity of the diagonal coefficients) there exist two positive constants $c_0 > 0$ and $0 < \theta < 1$ such that

$$c_0 \frac{|\lambda|^2}{(1+|y^\alpha|)^\theta} \le \sum_{i,j=1}^n a_{i,j}^{\alpha,\alpha}(x,y)\lambda_i\lambda_j,$$

for almost all $x \in \Omega$, all $y \in \mathbb{R}^N$, all $\lambda \in \mathbb{R}^n$ and all $\alpha \in \{1,\dots,N\}$.

We note that, because of condition $(\mathcal{A}_3^*)$, the differential operator

$$-\sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\sum_{\beta=1}^N \sum_{j=1}^n a_{i,j}^{\alpha,\beta}(x,u(x))\frac{\partial u^\beta(x)}{\partial x_j}\right)$$

in (2.1) is not coercive on $W_0^{1,2}(\Omega)$.

For the case $N = 1$, that is, (2.1) is only one single degenerate elliptic equation, existence and regularity results have been deeply studied, we refer the reader to [16,17,43–50].

We now use Lemma 1.4 to prove the following:

Theorem 2.2. *Suppose u is a solution to (2.1) with respect to $f \in L^m(\Omega;\mathbb{R}^N)$, $m > (2^*)' = \frac{2n}{n+2}$. Under the assumptions $(\mathcal{A}_1)$, $(\mathcal{A}_2)$, $(\mathcal{A}_3^*)$ and $(\mathcal{A}_4)$, we have*

(1) if $\frac{2n}{n+2} < m < \frac{n}{2}$, then $|u| \in L_{weak}^{m^{**}(1-\theta)}(\Omega)$;

(2) if $m = \frac{n}{2}$, then for any $q > 1$, $|u| \in L_{weak}^{\tilde{q}}(\Omega)$, $\tilde{q} = \max\{q, 1+2^*\}$;

(3) if $m > \frac{n}{2}$, then there exists $\lambda > 0$ such that $e^{\lambda|u|^{\log_N \sqrt{\beta}}} \in L^1(\Omega)$.

Proof. In order to prove Theorem 2.2-(1), we take a test function $\varphi = (\varphi^1,\dots,\varphi^N)$ in (2.2) as

$$\varphi^\alpha = \sum_{\gamma=1}^N c_\alpha^\gamma T_k(G_k(u^\gamma)),\ \alpha \in \{1,\dots,N\}, \tag{2.26}$$

here c_α^γ, $\alpha,\gamma \in \{1,\dots,N\}$ are real constants to be chosen later. Then

$$D_i\varphi^\alpha = \sum_{\gamma=1}^N c_\alpha^\gamma D_i u^\gamma 1_{B_k^\gamma},$$

where

$$B_k^\gamma = \{x \in \Omega : k \le |u^\gamma| < 2k\},\quad \gamma = 1,\dots,N.$$

(2.2) with the test function φ as in (2.26) gives

$$\begin{aligned} &\int_\Omega \sum_{\alpha,\beta=1}^N \sum_{i,j=1}^n a_{i,j}^{\alpha,\beta} D_j u^\beta \sum_{\gamma=1}^N c_\alpha^\gamma D_i u^\gamma 1_{B_k^\gamma} dx \\ =\ &\sum_{\alpha=1}^N \int_\Omega f^\alpha \sum_{\gamma=1}^N c_\alpha^\gamma T_k(G_k(u^\gamma))dx, \end{aligned} \tag{2.27}$$

where we have denoted again $a_{i,j}^{\alpha,\beta} = a_{i,j}^{\alpha,\beta}(x,u)$.

For the left hand side of (2.27), we note that (by the same reason as (2.6), with A_k^β replaced by B_k^β and A_k^γ replaced by B_k^γ)

$$\begin{aligned}
&\int_\Omega \sum_{\alpha,\beta=1}^N \sum_{i,j=1}^n a_{i,j}^{\alpha,\beta} D_j u^\beta \sum_{\gamma=1}^N c_\alpha^\gamma D_i u^\gamma 1_{B_k^\gamma} dx \\
= &\int_\Omega \sum_{\alpha=1}^N \sum_{i,j=1}^n c_\alpha^\alpha a_{i,j}^{\alpha,\alpha} D_j u^\alpha D_i u^\alpha 1_{B_k^\alpha} dx \\
&+ \int_\Omega \sum_{\alpha=1}^N \sum_{i,j=1}^n a_{i,j}^{\alpha,\alpha} D_j u^\alpha \sum_{\gamma=1,\gamma\neq\alpha}^N c_\alpha^\gamma D_i u^\gamma 1_{B_k^\gamma} dx \\
&+ \int_\Omega \sum_{\alpha=1}^N \sum_{\beta=1,\beta\neq\alpha}^N \sum_{i,j=1}^n r^{\alpha,\beta} a_{i,j}^{\beta,\beta} D_j u^\beta c_\alpha^\beta D_i u^\beta 1_{B_k^\beta} dx \\
&+ \int_\Omega \sum_{\alpha=1}^N \sum_{\beta=1,\beta\neq\alpha}^N \sum_{i,j=1}^n r^{\alpha,\beta} a_{i,j}^{\beta,\beta} D_j u^\beta \sum_{\gamma=1,\gamma\neq\beta}^N c_\alpha^\gamma D_i u^\gamma 1_{B_k^\gamma} dx \\
= &\tilde{I}_1 + \tilde{I}_2 + \tilde{I}_3 + \tilde{I}_4.
\end{aligned} \tag{2.28}$$

It is obvious that

$$\tilde{I}_1 + \tilde{I}_3 = \int_\Omega \sum_{\beta=1}^N \sum_{i,j=1}^n \left(\sum_{\alpha=1}^N r^{\alpha,\beta} c_\alpha^\beta \right) a_{i,j}^{\beta,\beta} D_j u^\beta D_i u^\beta 1_{B_k^\beta} dx \tag{2.29}$$

and

$$\tilde{I}_2 + \tilde{I}_4 = \int_{B_k^\gamma} \sum_{\beta,\gamma=1,\beta\neq\gamma}^N \sum_{i,j=1}^n \left(\sum_{\alpha=1}^N r^{\alpha,\beta} c_\alpha^\gamma \right) a_{i,j}^{\beta,\beta} D_j u^\beta D_i u^\gamma dx. \tag{2.30}$$

As in the proof of Theorem 2.1, by the condition $(\mathcal{A}_4)$, one can choose

$$\sum_{\alpha=1}^N r^{\alpha,\beta} c_\alpha^\beta = 1, \quad \text{for } \beta \in \{1, \dots, N\} \tag{2.31}$$

and

$$\sum_{\alpha=1}^N r^{\alpha,\beta} c_\alpha^\gamma = 0, \quad \text{for } \beta,\gamma \in \{1, \dots, N\}, \beta \neq \gamma, \tag{2.32}$$

then (2.31) and $(\mathcal{A}_3^*)$ allow us to estimate

$$\tilde{I}_1 + \tilde{I}_3 = \int_\Omega \sum_{\beta=1}^N \sum_{i,j=1}^n a_{i,j}^{\beta,\beta} D_j u^\beta D_i u^\beta 1_{B_k^\beta} dx \geq c_0 \sum_{\beta=1}^N \int_{B_k^\beta} \frac{|Du^\beta|^2}{(1+|u^\beta|)^\theta} dx, \tag{2.33}$$

and (2.30) together with (2.32) implies

$$\tilde{I}_2 + \tilde{I}_4 = 0. \tag{2.34}$$

Combining (2.27), (2.28), (2.33) and (2.34) one has

$$c_0 \sum_{\beta=1}^N \int_{B_k^\beta} \frac{|Du^\beta|^2}{(1+|u^\beta|)^\theta} dx \leq \sum_{\alpha=1}^N \int_\Omega f^\alpha \sum_{\gamma=1}^N c_\alpha^\gamma T_k(G_k(u^\gamma)) dx. \tag{2.35}$$

We now estimate both sides of (2.35). For the left hand side, since $1+|u^\beta|<1+2k$ for $x\in B_k^\beta$, then we use Sobolev Embedding Theorem again in order to derive

$$\begin{aligned} & c_0\sum_{\beta=1}^N\int_{B_k^\beta}\frac{|Du^\beta|^2}{(1+|u^\beta|)^\theta}dx \\ \geq\ & \frac{c_0}{(1+2k)^\theta}\sum_{\beta=1}^N\int_{B_k^\beta}|Du^\beta|^2dx \\ =\ & \frac{c_0}{(1+2k)^\theta}\sum_{\beta=1}^N\int_\Omega|DT_k(G_k(u^\beta))|^2dx \\ \geq\ & \frac{c_0}{c_*^2(1+2k)^\theta}\sum_{\beta=1}^N\left(\int_\Omega|T_k(G_k(u^\beta))|^{2^*}dx\right)^{\frac{2}{2^*}} \\ \geq\ & \frac{c_0}{c_*^2(1+2k)^\theta}\sum_{\beta=1}^N\left(\int_{A_{2k}^\beta}|T_k(G_k(u^\beta))|^{2^*}dx\right)^{\frac{2}{2^*}} \\ =\ & \frac{c_0k^2}{c_*^2(1+2k)^\theta}\sum_{\beta=1}^N|A_{2k}^\beta|^{\frac{2}{2^*}}. \end{aligned} \tag{2.36}$$

For the right hand side, since $G_k(u^\gamma)=0$ for $x\in\{x\in\Omega:|u^\gamma(x)|\leq k\}$ and $|T_k(G_k(u^\gamma))|\leq k$, then the condition $f\in L^m(\Omega;\mathbb{R}^N)$ $(\frac{2n}{n+2}<m<\frac{n}{2})$ together with Hölder inequality implies

$$\begin{aligned} & \sum_{\alpha=1}^N\int_\Omega f^\alpha\sum_{\gamma=1}^N c_\alpha^\gamma T_k(G_k(u^\gamma))dx \\ =\ & \sum_{\gamma=1}^N\int_{A_k^\gamma}\sum_{\alpha=1}^N c_\alpha^\gamma f^\alpha T_k(G_k(u^\gamma))dx \\ \leq\ & \sum_{\gamma=1}^N\int_{A_k^\gamma}\left|\sum_{\alpha=1}^N c_\alpha^\gamma f^\alpha\right||T_k(G_k(u^\gamma))|\,dx \\ \leq\ & k\sum_{\gamma=1}^N\int_{A_k^\gamma}\left|\sum_{\alpha=1}^N c_\alpha^\gamma f^\alpha\right|dx \\ \leq\ & k\sum_{\gamma=1}^N\left(\int_{A_k^\gamma}\left|\sum_{\alpha=1}^N c_\alpha^\gamma f^\alpha\right|^m dx\right)^{\frac{1}{m}}|A_k^\gamma|^{\frac{1}{m'}} \\ \leq\ & c_{20}k\sum_{\gamma=1}^N|A_k^\gamma|^{\frac{1}{m'}}, \end{aligned} \tag{2.37}$$

where

$$c_{20}=\sum_{\gamma=1}^N\left(\int_\Omega\left|\sum_{\alpha=1}^N c_\alpha^\gamma f^\alpha\right|^m dx\right)^{\frac{1}{m}}.$$

Substituting (2.36) and (2.37) into (2.35) one has

$$\sum_{\beta=1}^N|A_{2k}^\beta|^{\frac{2}{2^*}}\leq\frac{c_{20}c_*^2(1+2k)^\theta}{c_0k}\sum_{\gamma=1}^N|A_k^\gamma|^{\frac{1}{m'}}.$$

For all $k\geq k_0=1$, $(1+2k)^\theta\leq(3k)^\theta$, then

$$\sum_{\beta=1}^N|A_{2k}^\beta|^{\frac{2}{2^*}}\leq\frac{c_{20}c_*^2 3^\theta}{c_0k^{1-\theta}}\sum_{\gamma=1}^N|A_k^\gamma|^{\frac{1}{m'}}=\frac{c_{21}}{k^{1-\theta}}\sum_{\gamma=1}^N|A_k^\gamma|^{\frac{1}{m'}}, \tag{2.38}$$

where $c_{21} = \frac{c_{20}c_*^2 3^\theta}{c_0}$. We recall that $A_k = \{x \in \Omega : |u(x)| > k\}$ and (2.20), from the above inequality we derive

$$|A_{2Nk}|^{\frac{2}{2^*}} \le \left(\sum_{\beta=1}^N |A_{2k}^\beta|\right)^{\frac{2}{2^*}} \le \sum_{\beta=1}^N |A_{2k}^\beta|^{\frac{2}{2^*}}$$
$$\le \frac{c_{21}}{k^{1-\theta}} \sum_{\gamma=1}^N |A_k^\gamma|^{\frac{1}{m'}} \le \frac{c_{21}N}{k^{1-\theta}} |A_k|^{\frac{1}{m'}},$$

which is equivalent to

$$|A_{2Nk}| \le \frac{(c_{21}N)^{\frac{2^*}{2}}}{k^{\frac{(1-\theta)2^*}{2}}} |A_k|^{\frac{2^*}{2m'}}.$$

The inequality (∗) in Remark 1.1 holds with

$$\varphi(k) = |A_k|,\ \tilde{c}_3 = (c_{21}N)^{\frac{2^*}{2}},\ \alpha = \frac{(1-\theta)2^*}{2} \quad \text{and} \quad \beta = \frac{2^*}{2m'}.$$

Since $m < \frac{n}{2}$, then $0 < \beta < 1$. We are now in a position to use Remark 1.1 which states that the inequality (∗) is equivalent to (1.3), then we use Lemma 1.2 to derive that

$$|A_k| \le c_4 \left(\frac{1}{k}\right)^{\frac{\alpha}{1-\beta}} = c_4 \left(\frac{1}{k}\right)^{m^{**}(1-\theta)},$$

which is equivalent to $|u| \in L_{weak}^{m^{**}(1-\theta)}(\Omega)$, as desired.

In order to prove Theorem 2.2-(2), (3), we take a test function $\varphi = (\varphi^1, \ldots, \varphi^N)$ in (2.2) with φ^α as in (2.4). We follow the lines of the proof of Theorem 2.1 until we arrive at (2.10). For appropriate choice of the constants C_α^γ, $\alpha, \gamma \in \{1, \ldots, N\}$, (2.9) and (2.10) can be satisfied. Then (2.9) and $(\mathcal{A}_3^*)$ allow us to estimate

$$I_1 + I_3 = \int_\Omega \sum_{\beta=1}^N \sum_{i,j=1}^n a_{i,j}^{\beta,\beta} D_j u^\beta D_i u^\beta 1_{A_k^\beta} dx \ge c_0 \sum_{\beta=1}^N \int_{A_k^\beta} \frac{|Du^\beta|^2}{(1+|u^\beta|)^\theta} dx. \tag{2.39}$$

Combining (2.5), (2.6), (2.39) and (2.12) one has

$$\begin{aligned} c_0 \sum_{\beta=1}^N \int_{A_k^\beta} \frac{|Du^\beta|^2}{(1+|u^\beta|)^\theta} dx &\le \sum_{\alpha=1}^N \int_\Omega f^\alpha \sum_{\gamma=1}^N C_\alpha^\gamma G_k(u^\gamma) dx \\ = \sum_{\beta=1}^N \int_\Omega \left(\sum_{\alpha=1}^N C_\alpha^\beta f^\alpha\right) G_k(u^\beta) dx &\le c_{22} \sum_{\beta=1}^N \left(\int_{A_k^\beta} |G_k(u^\beta)|^{m'} dx\right)^{\frac{1}{m'}}, \end{aligned} \tag{2.40}$$

where $c_{22} = \left\| \sum_{\alpha=1}^N C_\alpha^\beta f^\alpha \right\|_{L^m(\Omega)}$. Let

$$\sigma = \frac{nm}{nm - n + m},$$

the condition $m > (2^*)'$ implies $\sigma < 2$. We can write, by Hölder inequality and (2.40),

$$\begin{aligned} &\sum_{\beta=1}^N \int_{A_k^\beta} |Du^\beta|^\sigma dx \\ = &\sum_{\beta=1}^N \int_{A_k^\beta} \frac{|Du^\beta|^\sigma}{(1+|u^\beta|)^{\frac{\theta\sigma}{2}}} (1+|u^\beta|)^{\frac{\theta\sigma}{2}} dx \\ \le &\sum_{\beta=1}^N \left(\int_{A_k^\beta} \frac{|Du^\beta|^2}{(1+|u^\beta|)^\theta} dx\right)^{\frac{\sigma}{2}} \left(\int_{A_k^\beta} (1+|u^\beta|)^{\frac{\theta\sigma}{2-\sigma}} dx\right)^{\frac{2-\sigma}{2}} \\ \le &\left(\sum_{\beta=1}^N \int_{A_k^\beta} \frac{|Du^\beta|^2}{(1+|u^\beta|)^\theta} dx\right)^{\frac{\sigma}{2}} \sum_{\beta=1}^N \left(\int_{A_k^\beta} (1+|u^\beta|)^{\frac{\theta\sigma}{2-\sigma}} dx\right)^{\frac{2-\sigma}{2}}. \end{aligned} \tag{2.41}$$

Note that $\sigma^* = m'$, then Sobolev inequality allows us to estimate

$$\begin{aligned}
&\left(\sum_{\beta=1}^N \int_{A_k^\beta} \frac{|Du^\beta|^2}{(1+|u^\beta|)^\theta} dx\right)^{\frac{\sigma}{2}} \\
\leq\ & \left[\frac{c_{22}}{c_0} \sum_{\beta=1}^N \left(\int_{A_k^\beta} |G_k(u^\beta)|^{m'} dx\right)^{\frac{1}{m'}} dx\right]^{\frac{\sigma}{2}} \\
\leq\ & \left(\frac{c_{22}}{c_0}\right)^{\frac{\sigma}{2}} \sum_{\beta=1}^N \left(\int_\Omega |G_k(u^\beta)|^{\sigma^*} dx\right)^{\frac{1}{\sigma^*}\frac{\sigma}{2}} \\
\leq\ & c_* \left(\frac{c_{22}}{c_0}\right)^{\frac{\sigma}{2}} \sum_{\beta=1}^N \left(\int_\Omega |DG_k(u^\beta)|^{\sigma} dx\right)^{\frac{1}{2}} \\
\leq\ & c_* 2^N \left(\frac{c_{22}}{c_0}\right)^{\frac{\sigma}{2}} \left(\sum_{\beta=1}^N \int_{A_k^\beta} |Du^\beta|^{\sigma} dx\right)^{\frac{1}{2}}.
\end{aligned} \tag{2.42}$$

Substituting (2.42) into (2.41) we arrive at

$$\sum_{\beta=1}^N \int_{A_k^\beta} |Du^\beta|^\sigma dx \leq c_{23} \left[\sum_{\beta=1}^N \left(\int_{A_k^\beta} (1+|u^\beta|)^{\frac{\theta\sigma}{2-\sigma}} dx\right)^{\frac{2-\sigma}{2}}\right]^2, \tag{2.43}$$

where $c_{23} = \left(c_* 2^N\right)^2 \left(\frac{c_{22}}{c_0}\right)^\sigma$. If $k \geq k_0 = 1$, one has on A_k^β that $1+|u^\beta| \leq 2(k+|G_k(u^\beta)|)$, then

$$\begin{aligned}
&\sum_{\beta=1}^N \int_{A_k^\beta} |Du^\beta|^\sigma dx \\
\leq\ & c_{23} \left[\sum_{\beta=1}^N \left(\int_{A_k^\beta} (2(k+|G_k(u^\beta)|))^{\frac{\theta\sigma}{2-\sigma}} dx\right)^{\frac{2-\sigma}{2}}\right]^2 \\
\leq\ & c_{24} \left[\sum_{\beta=1}^N \left(\int_{A_k^\beta} k^{\frac{\theta\sigma}{2-\sigma}} + |G_k(u^\beta)|^{\frac{\theta\sigma}{2-\sigma}} dx\right)^{\frac{2-\sigma}{2}}\right]^2 \\
\leq\ & c_{25} \sum_{\beta=1}^N \left[k^{\theta\sigma}|A_k^\beta|^{2-\sigma} + \left(\int_{A_k^\beta} |G_k(u^\beta)|^{\frac{\theta\sigma}{2-\sigma}} dx\right)^{2-\sigma}\right].
\end{aligned} \tag{2.44}$$

Since

$$\theta < 1,\ m \geq \frac{n}{2} \Rightarrow \frac{\theta\sigma}{2-\sigma} < \sigma^*,$$

then using Hölder, Sobolev and Young inequalities, one obtains

$$\begin{aligned}
&\sum_{\beta=1}^N \int_{A_k^\beta} |Du^\beta|^\sigma dx \\
\leq\ & c_{25} \sum_{\beta=1}^N \left[k^{\theta\sigma}|A_k^\beta|^{2-\sigma} + \left(\int_{A_k^\beta} |G_k(u^\beta)|^{\sigma^*} dx\right)^{\frac{\theta\sigma}{\sigma^*}} |A_k^\beta|^{2-\sigma-\frac{\theta\sigma}{\sigma^*}}\right] \\
\leq\ & c_{25} \sum_{\beta=1}^N \left[k^{\theta\sigma}|A_k^\beta|^{2-\sigma} + c_* \left(\int_{A_k^\beta} |Du^\beta|^{\sigma} dx\right)^{\theta} |A_k^\beta|^{2-\sigma-\frac{\theta\sigma}{\sigma^*}}\right] \\
\leq\ & c_{25} \sum_{\beta=1}^N \left[k^{\theta\sigma}|A_k^\beta|^{2-\sigma} + \varepsilon \int_{A_k^\beta} |Du^\beta|^{\sigma} dx + c(\varepsilon)|A_k^\beta|^{\frac{(2-\sigma)\sigma^*-\theta\sigma}{(1-\theta)\sigma^*}}\right].
\end{aligned} \tag{2.45}$$

If we choose ε small enough such that $c_{25}N\varepsilon = \frac{1}{2}$, then the second term in the right hand side of the above inequality can be absorbed by the left hand side, then

$$\sum_{\beta=1}^{N}\int_{A_k^\beta}|Du^\beta|^\sigma dx \le c_{26}\sum_{\beta=1}^{N}\left[k^{\theta\sigma}|A_k^\beta|^{2-\sigma} + |A_k^\beta|^{\frac{(2-\sigma)\sigma^*-\theta\sigma}{(1-\theta)\sigma^*}}\right].$$

$m \ge \frac{n}{2}$ implies

$$2-\sigma \le \frac{(2-\sigma)\sigma^*-\theta\sigma}{(1-\theta)\sigma^*},$$

so that, for $k \ge k_0 = 1$, we can write, observing that $|A_k^\beta| \le |\Omega|$,

$$\sum_{\beta=1}^{N}\int_{A_k^\beta}|Du^\beta|^\sigma dx \le c_{27}\sum_{\beta=1}^{N}k^{\theta\sigma}|A_k^\beta|^{2-\sigma} \le c_{27}k^{\theta\sigma}\sum_{\beta=1}^{N}|A_k|^{2-\sigma}. \tag{2.46}$$

The left hand side of the above inequality can be estimated as follows: for any $h > k \ge k_0 = 1$,

$$\begin{aligned}
&\sum_{\beta=1}^{N}\int_{A_k^\beta}|Du^\beta|^\sigma dx = \sum_{\beta=1}^{N}\int_{\Omega}|DG_k(u^\beta)|^\sigma dx \\
\ge\ & \frac{1}{c_*}\sum_{\beta=1}^{N}\left(\int_{A_k^\beta}|G_k(u^\beta)|^{\sigma^*}dx\right)^{\frac{\sigma}{\sigma^*}} \ge \frac{1}{c_*}\sum_{\beta=1}^{N}\left(\int_{A_h^\beta}|G_k(u^\beta)|^{\sigma^*}dx\right)^{\frac{\sigma}{\sigma^*}} \\
=\ & \frac{1}{c_*}(h-k)^\sigma\sum_{\beta=1}^{N}|A_h^\beta|^{\frac{\sigma}{\sigma^*}}.
\end{aligned} \tag{2.47}$$

Combining (2.46) and (2.47) and making use of (2.20), we have

$$|A_{Nh}|^{\frac{\sigma}{\sigma^*}} \le \left(\sum_{\beta=1}^{N}|A_h^\beta|\right)^{\frac{\sigma}{\sigma^*}} \le \sum_{\beta=1}^{N}|A_h^\beta|^{\frac{\sigma}{\sigma^*}} \le \frac{c_*c_{27}k^{\theta\sigma}}{(h-k)^\sigma}\sum_{\beta=1}^{N}|A_k|^{2-\sigma},$$

from which we derive

$$|A_{Nh}| \le \frac{c_{28}k^{\theta\sigma^*}}{(h-k)^{\sigma^*}}|A_k|^{\frac{(2-\sigma)\sigma^*}{\sigma}}.$$

The condition (1.22) in Lemma 1.4 holds true with

$$k_0 = 1, \varphi(k) = |A_k|, c_{10} = c_{28}, \alpha = \sigma^*,\ \text{and}\ \beta = \frac{(2-\sigma)\sigma^*}{\sigma}.$$

Now we use Lemma 1.4 to derive that:

If $m = \frac{n}{2}$, then $\beta = 1$, thus for any $q > 1$,

$$|A_k| \le c_{12}\left(\frac{1}{k}\right)^{\tilde{q}},\ \tilde{q} = \max\{q, 1+\sigma^*\},$$

which is equivalent to

$$|u| \in L^{\tilde{q}}_{weak}(\Omega).$$

If $m > \frac{n}{2}$, then $\beta > 1$. As in the proof of part (3) in Theorem 2.1 one can derive the desired result $e^{\lambda|u|^{\log_N\sqrt{\beta}}} \in L^1(\Omega)$ for some $\lambda > 0$. □

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