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Group decision making based on multiplicative consistency-and-consensus preference analysis for incomplete q-rung orthopair fuzzy preference relations

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ABSTRACT

The q-rung orthopair fuzzy preference relations are useful tools to represent hesitant and uncertain judgments of decision makers. In this paper, we propose a new group decision making method based on multiplicative consistency-and-consensus preference analysis for incomplete q-rung orthopair fuzzy preference relations. First, we provide a novel concept of multiplicative consistency for q-rung orthopair fuzzy preference relations. Then, a multiplicative consistency index is offered, by which we derive the concept of acceptable multiplicative consistency for q-rung orthopair fuzzy preference relations. Following this concept, optimization models for ascertaining unknown values in an incomplete q-rung orthopair fuzzy preference relation are built. Furthermore, optimization models for obtaining acceptable multiplicative q-rung orthopair fuzzy preference relation are proposed. Then, an optimization model for group decision making is proposed to attain an enough consensus. Afterward, a group decision making method with incomplete and unacceptable multiplicative consistent q-rung orthopair fuzzy preference relations is proposed. Finally, we use an application example to show the practicality of the proposed group decision making method. The proposed group decision making method outperforms the existing group decision making methods for group decision making in incomplete q-rung orthopair fuzzy environments.

1. Introduction

Group decision making (GDM) is an attractive and useful process to deal with complex decision situations [10,11,13,17,19,22–24,29,39,41,42,44,46], where several experts get together to make a group decision. In various GDM methods, preference relations are efficient and relatively simple tools, whose goal is to obtain the ranking order of alternatives based on pairwise judgments between alternatives. Fuzzy preference relations (FPRs) [20] and multiplicative preference relations (MPRs) [26] are two kinds of basic and useful preference relations. Extensions of these two kinds of preference relations have been proposed, including intuitionistic FPRs [36], intuitionistic MPRs [33], interval FPRs [35], interval MPRs [27], hesitant FPRs [50] and hesitant MPRs [49], . . ., etc. Recently, Yager [37] presented the notion of q-rung ortho-

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pair fuzzy sets (q-ROFSs), whose membership degree $\mu_R(u)$ and non-membership degree $v_R(u)$ satisfy the restriction $0 \leqslant \mu_R^q(u) + v_R^q(u) \leqslant 1$, where $q \geqslant 1$. Clearly, if $q = 1$, then q-ROFSs are a generalized form of intuitionistic fuzzy sets [5]; if $q = 2$, then q-ROFSs are a generalized form of Pythagorean fuzzy sets [38]. Thus, q-ROFSs are more general and more flexible than intuitionistic fuzzy sets and Pythagorean fuzzy sets. Up to now, q-ROFSs have been widely applied to deal with decision making problems. Many aggregation operators in the environment of q-ROFSs have been developed [6,7,9,12,14,15,16,21]. Moreover, Zhang et al. [47] presented the notion of q-rung orthopair fuzzy preference relations (q-ROFPRs) based on q-ROFSs and preference relations. For the concept of additive consistency for q-ROFPRs, it was originally proposed in [47]. Based on the concept of additive consistency of q-ROFPRs, Zhang et al. [47] established a linear programming model to obtain the q-rung orthopair fuzzy (q-ROF) priority weight vector from a q-ROFPR or a set of q-ROFPRs. For an unacceptable consistent q-ROFPR, some goal programming models are built in [45] for improving its consistency level. For the concept of multiplicative consistency of q-ROFPRs, it was originally proposed in [48]. Following the definition of the consistency of q-ROFPRs, Zhang et al. [48] presented a programming model to derive the normalized q-ROF priority weights from a q-ROFPR. Furthermore, Zhang et al. [48] established a goal programming method to determine the priority weights of alternatives from a set of q-ROFPRs. Moreover, Zhang et al. [48] offered an algorithm to repair the inconsistency of a q-ROFPR and proposed a GDM method with q-ROFPRs.

However, there are some drawbacks in the existing GDM methods [47,48], described as follows:

(1) As the GDM environment becomes more and more complicated, there are too many uncertain factors to make it impossible to obtain the complete evaluation information. Hence, it is needed to deal with incomplete information in GDM processes. However, the existing GDM methods [47,48] do not deal with incomplete q-ROFPRs, where an incomplete q-ROFPR contains unknown q-ROFNs.
(2) To make a reasonable GDM, a group of decision makers (DMs) are usually involved in practice. Thus, how to reach an agreement among DMs whose perceptions are represented by q-ROFPRs is very important. To meet this challenge, the consensus among the opinions of DMs is an important topic for GDM with q-ROFPRs, where a consensus procedure is worth to be considered during GDM processes. However, the GDM methods presented in Zhang et al. [47] and [48] do not consider this aspect.
(3) Based on the definition of q-ROFPR, Zhang et al. [47] discussed its additive consistency, which relates to the validity and the dependability of the priority weights of alternatives. However, the consistency test and the adjustment of q-ROFPRs are not considered in the method presented in Zhang et al. [47].
(4) In GDM problems, ignoring the influence of DMs' weights will lead to unreasonable ranking orders of alternatives. Thus, we must consider the influence of DMs' weights in GDM processes. In Zhang et al.'s GDM method [47], the weights of DMs are predefined, which are subjective and artificial.
(5) In order to guarantee the consistency of q-ROFPRs is reached, Zhang et al.'s GDM method [48] presented an adjustment procedure to let an original q-ROFPR becomes a consistent q-ROFPR. With the consistency improving algorithm presented in Zhang et al. [48], it needs to derive the consistent q-ROFPR of an original q-ROFPR at each iteration, which makes GDM processes more complicated. Furthermore, Zhang et al.'s GDM method [48] needed to adjust most of the original judgements in q-ROFPRs, which means that most of the original evaluation information in q-ROFPRs may be lost.

In order to effectively overcome the above drawbacks, we deal with incomplete q-ROFPRs from the perspective of consistency and the consensus to propose a new GDM method. The main contributions of this paper are highlighted as follows:

(1) Different from the definition of additive consistency presented in Zhang and Chen [45] for q-ROFPRs, we propose a multiplicative consistency concept. Following the proposed consistency concept of q-ROFPR, we propose a consistency index for a q-ROFPR. Then, the definition of an acceptable consistent q-ROFPR is defined. A model for estimating unknown values in an incomplete q-ROFPR is proposed through minimizing the consistency index. Meanwhile, a repairing model is offered to adjust an unsatisfactorily consistent q-ROFPR to obtain an acceptable multiplicative consistent q-ROFPR. Subsequently, we propose a new individual decision making method called "Algorithm 1" with an incomplete q-ROFPR.
(2) Based on the definition of correlation coefficient, we propose an approach to determine the weights of DMs in GDM processes. Furthermore, it is proven that if each individual q-ROFPR is acceptable consistent, then the collective q-ROFPR is acceptable consistent.
(3) Through introducing a group consensus index for a set of DMs, we introduce a way to improve their consensus when DMs are unsatisfied with their current consensus degree. Then, we propose a GDM method called "Algorithm 2" based on incomplete q-ROFPRs.
(4) We utilize the proposed GDM method to deal with an application example of selecting the best paper and provide some comparative analyses to exhibit the merits of the proposed GDM method.

The rest of this paper is organized as follows. Section 2 presents the preliminaries of this paper. Section 3 makes the acceptable multiplicative consistency analysis and proposes a new individual decision making method with an incomplete

q-ROFPR. Section 4 presents a method to determine DMs' weights and proposes a new GDM method, called **Algorithm 2**, based on the multiplicative consistency-and-consensus preference analysis of incomplete q-ROFPRs. Section 5 offers an example of selecting the best paper. Section 6 presents the conclusions.

2. Preliminaries

As the basis of this paper, we review several related concepts. Yager [37] presented the concept of q-rung orthopair fuzzy sets (q-ROFSs). A q-ROFS R is represented by $R = \{\langle u, \mu_R(u), v_R(u)\rangle | u \in U\}$, where U is an ordinary fixed set, the membership function $\mu_R : U \to [0,1]$ and the non-membership function $v_R : U \to [0,1]$ define the membership degree and the non-membership degree of element $u \in U$ belonging to the q-ROFS R, respectively, where for each $u \in U$, $0 \leqslant \mu_R^q(u) + v_R^q(u) \leqslant 1$ and $q \geqslant 1$.

A q-rung orthopair fuzzy number (q-ROFN) [14] is represented by $\alpha = (\mu_\alpha, v_\alpha)$, where $\mu_\alpha, v_\alpha \in [0,1]$, $0 \leqslant \mu_\alpha^q + v_\alpha^q \leqslant 1$ and $q \geqslant 1$.

Definition 2.1 [14]. Let $\alpha = (\mu_\alpha, v_\alpha)$ be a q-ROFN. The score value $S(\alpha)$ of the q-ROFN $\alpha = (\mu_\alpha, v_\alpha)$ is defined by $S(\alpha) = \mu_\alpha^q - v_\alpha^q$, where $q \geqslant 1$; the accuracy value $H(\alpha)$ of the q-ROFN $\alpha = (\mu_\alpha, v_\alpha)$ is defined by $H(\alpha) = \mu_\alpha^q + v_\alpha^q$, where S and H are the score function and the accuracy function of q-ROFNs, respectively, and $q \geqslant 1$.

Let $\alpha_1 = \left(\mu_{\alpha_1}, v_{\alpha_1}\right)$ and $\alpha_2 = \left(\mu_{\alpha_2}, v_{\alpha_2}\right)$ be two q-ROFNs. According to [14], we can see that

(1) If $S(\alpha_1) > S(\alpha_2)$, then $\alpha_1 > \alpha_2$;
(2) If $S(\alpha_1) = S(\alpha_2)$, then

If $H(\alpha_1) > H(\alpha_2)$, then $\alpha_1 > \alpha_2$;
If $H(\alpha_1) = H(\alpha_2)$, then $\alpha_1 = \alpha_2$.

Definition 2.2 [47]. Let $X = \{x_1, x_2, \cdots, x_n\}$ be a set of alternatives and let the q-ROFN $\alpha_{ij} = \left(\mu_{ij}, v_{ij}\right)$ denote the degree that alternative x_i prefers to alternative x_j, where $i = 1,2,\cdots,n$, $j = 1,2,\cdots,n$ and $q \geqslant 1$. The matrix $A = (\alpha_{ij})_{n\times n}$ consisting of $n \times n$ q-ROFNs is called a q-ROFPR, which satisfies:

$$\begin{cases} \mu_{ij}, v_{ij} \in [0,1], \\ \mu_{ij} = v_{ji}, \\ v_{ij} = \mu_{ji}, \\ \mu_{ii} = v_{ii} = \sqrt[q]{0.5}, \\ 0 \leqslant \mu_{ij}^q + v_{ij}^q \leqslant 1, \end{cases} \tag{1}$$

where $i = 1,2,\cdots,n$, $j = 1,2,\cdots,n$ and $q \geqslant 1$.

Zhang *et al.* [48] introduced the following multiplicative consistency concept of q-ROFPRs.

Definition 2.3 [48]. Let $A = (\alpha_{ij})_{n\times n}$ be a q-ROFPR, where $\alpha_{ij} = \left(\mu_{ij}, v_{ij}\right)$ is a q-ROFN, $i = 1,2,\cdots,n$, $j = 1,2,\cdots,n$ and $q \geqslant 1$. The q-ROFPR $A = (\alpha_{ij})_{n\times n}$ is multiplicative consistent if the following is true:

$$\mu_{ij}\mu_{jk}\mu_{ki} = \mu_{ik}\mu_{kj}\mu_{ji} \tag{2}$$

for all $i, j, k = 1,2,\cdots,n$.

3. Acceptable multiplicative consistency of *q*-ROFPRs

In this section, we deal with incomplete and unacceptable multiplicatively consistent q-ROFPRs. First, a new multiplicative consistency definition for q-ROFPRs is presented. Then, a multiplicative consistency index for q-ROFPRs is offered. Subsequently, a multiplicative consistency-based optimization model for obtaining unknown values of incomplete q-ROFPRs is proposed. Afterward, when q-ROFPRs are unacceptable multiplicatively consistent, a procedure for reaching the acceptable multiplicative consistency requirement is proposed. Moreover, a programming model to get the q-POF priority vector is built. Finally, a decision making method including a q-ROFPR is presented.

3.1. Multiplicative consistency of q-ROFPRs

Based on **Definition 2.2** and **Definition 2.3**, we can deduce the following property of q-ROFNs.

Theorem 3.1. *Let $A = (\alpha_{ij})_{n\times n}$ be a q-ROFPR, where $\alpha_{ij} = \left(\mu_{ij}, v_{ij}\right)$ is a q-ROFN, $i = 1,2,\cdots,n$, $j = 1,2,\cdots,n$ and $q \geqslant 1$. The following statements are equivalent*:

(1) $\mu_{ij}\mu_{jk}\mu_{ki} = \mu_{ik}\mu_{kj}\mu_{ji}$, for all $i,j,k = 1,2,\cdots,n$;
(2) $v_{ij}v_{jk}v_{ki} = v_{ik}v_{kj}v_{ji}$, for all $i,j,k = 1,2,\cdots,n$;
(3) $\mu_{ij}\mu_{jk}\mu_{ki} = v_{ij}v_{jk}v_{ki}$, for all $i,j,k = 1,2,\cdots,n$;
(4) $\mu_{ij}\mu_{jk}\mu_{ki} = \mu_{ik}\mu_{kj}\mu_{ji}$, for all $i,j,k = 1,2,\cdots,n$ with $i<j<k$;
(5) $v_{ij}v_{jk}v_{ki} = v_{ik}v_{kj}v_{ji}$, for all $i,j,k = 1,2,\cdots,n$ with $i<j<k$;
(6) $\mu_{ij}\mu_{jk}\mu_{ki} = v_{ij}v_{jk}v_{ki}$, for all $i,j,k = 1,2,\cdots,n$ with $i<j<k$;
(7) $\mu_{ij}\mu_{jk}v_{ik} = v_{ij}v_{jk}\mu_{ik}$, for all $i,j,k = 1,2,\cdots,n$ with $i<j<k$.

Based on **Definition 2.3** and **Theorem 3.1**, we propose the definition of multiplicative consistency of q-ROFNs.

Definition 3.1. Let $A = (\alpha_{ij})_{n\times n}$ be a q-ROFPR, where $\alpha_{ij} = \left(\mu_{ij}, v_{ij}\right)$ is a q-ROFN, $i = 1,2,\cdots,n, j = 1,2,\cdots,n$ and $q \geqslant 1$. The q-ROFPR $A = (\alpha_{ij})_{n\times n}$is multiplicative consistent if and only if

$$\mu_{ij}\mu_{jk}v_{ik} = v_{ij}v_{jk}\mu_{ik} \tag{3}$$

for all $i,j,k = 1,2,\cdots,n$ with $i<j<k$.

To measure the consistency of q-ROFPRs, it is urgent to provide a reliable consistency index and the corresponding threshold value.

Definition 3.2. Let $A = (\alpha_{ij})_{n\times n}$ be a q-ROFPR, where $\alpha_{ij} = \left(\mu_{ij}, v_{ij}\right)$ is a q-ROFN, $i = 1,2,\cdots,n, j = 1,2,\cdots,n$ and $q \geqslant 1$. The multiplicative consistency index $MCI(A)$ of the q-ROFPR $A = (\alpha_{ij})_{n\times n}$, where $\alpha_{ij} = \left(\mu_{ij}, v_{ij}\right)$ is a q-ROFN, is defined as follows:

$$MCI(A) = \frac{6}{n(n-1)(n-2)}\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n}\left|\ln\left(\mu_{ij}\right) + \ln\left(\mu_{jk}\right) + \ln(v_{ik}) - \ln(v_{ij}) - \ln(v_{jk}) - \ln(\mu_{ik})\right| \tag{4}$$

where $MCI(A) \geqslant 0$.

Because the q-ROFPRs given by DMs are usually unacceptable consistent, we must judge and improve their consistency levels. In the following, we use the proposed acceptable consistency index shown in **Definition 3.2** to discuss how to judge and derive acceptable consistent q-ROFPRs. The concept of acceptable multiplicative consistent q-ROFPRs is presented as follows.

Definition 3.3. Given a consistency threshold value δ, where $\delta \in [0,1]$, a q-ROFPR A is acceptable multiplicative consistent if $MCI(A) \leqslant \delta$, where $\delta \in [0,1]$.

3.2. Obtaining unknown values in incomplete q-ROFPRs

In some cases, we can only obtain incomplete q-ROFPRs, where some judgments in q-ROFPRs are missing. In order to deal with this situation, this subsection discusses incomplete q-ROFPRs and builds optimization models for estimating unknown q-ROFNs in incomplete q-ROFPRs. Let $A = (\alpha_{ij})_{n\times n}$ be an incomplete q-ROFPR, where $\alpha_{ij} = \left(\mu_{ij}, v_{ij}\right)$ is a q-ROFN, $i = 1,2,\cdots,n$, $j = 1,2,\cdots,n$ and $q \geqslant 1$, namely, there are some unknown values in the incomplete q-ROFPR $A = (\alpha_{ij})_{n\times n}$. Then, an auxiliary q-ROFPR $A' = \left(\alpha'_{ij}\right)_{n\times n}$ obtained from the incomplete q-ROFPR $A = (\alpha_{ij})_{n\times n}$ with $\alpha'_{ij} = \left(\mu'_{ij}, v'_{ij}\right)$ can be built, shown as follows:

$$\begin{aligned} \mu'_{ij} &= \begin{cases} \mu_{ij}, \text{ if } \mu_{ij} \text{ is known,} \\ \rho_{ij}, \text{ if } \mu_{ij} \text{ is unknown,} \end{cases} \\ v'_{ij} &= \begin{cases} v_{ij}, \text{ if } v_{ij} \text{ is known,} \\ \sigma_{ij}, \text{ if } v_{ij} \text{ is unknown.} \end{cases} \end{aligned} \tag{5}$$

where $0 \leqslant \rho_{ij}, \sigma_{ij} \leqslant 1,\ \rho_{ij}^q + \sigma_{ij}^q \leqslant 1, q \geqslant 1, i = 1,2,\cdots,n,$ and $j = 1,2,\cdots,n,$

Based on Eq. (5), we use the following model to estimate those missing judgments in the incomplete q-ROFPR A:

$$F = \min\ \xi$$

$$\text{s.t.}\begin{cases} \sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n}\left|\ln\left(\mu'_{ij}\right) + \ln\left(\mu'_{jk}\right) + \ln(v'_{ik}) - \ln\left(v'_{ij}\right) - \ln\left(v'_{jk}\right) - \ln(\mu'_{ik})\right| \leqslant \frac{n(n-1)(n-2)\xi}{6}, \\ 0 \leqslant \mu'_{ij} \leqslant 1,\ i,j = 1,2,\cdots,n,\ i<j, \\ 0 \leqslant v'_{ij} \leqslant 1,\ i,j = 1,2,\cdots,n,\ i<j, \\ \left(\mu'_{ij}\right)^q + \left(v'_{ij}\right)^q \leqslant 1,\ i,j = 1,2,\cdots,n,\ i<j. \end{cases} \tag{M-1}$$

After removing the symbol "| |" of the absolute value, the model (M-1) is transformed into:

$$F = \min \xi$$

$$s.t.\begin{cases} \sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n}\left(\zeta_{ijk}^{+}+\zeta_{ijk}^{-}\right) \leqslant \frac{n(n-1)(n-2)\xi}{6}, \\ \ln\left(\mu'_{ij}\right)+\ln\left(\mu'_{jk}\right)+\ln\left(\nu'_{ik}\right)-\ln\left(\nu'_{ij}\right)-\ln\left(\nu'_{jk}\right)-\ln\left(\mu'_{ik}\right)-\zeta_{ijk}^{+}+\zeta_{ijk}^{-}=0,\ i,j,k=1,2,\cdots,n,\ i<j<k, \\ \zeta_{ijk}^{+},\zeta_{ijk}^{-} \geqslant 0,\ i,j,k=1,2,\cdots,n,\ i<j<k. \\ 0 \leqslant \mu'_{ij} \leqslant 1,\ i,j=1,2,\cdots,n,\ i<j, \\ 0 \leqslant \nu'_{ij} \leqslant 1,\ i,j=1,2,\cdots,n,\ i<j, \\ \left(\mu'_{ij}\right)^{q}+\left(\nu'_{ij}\right)^{q} \leqslant 1,\ i,j=1,2,\cdots,n,\ i<j. \end{cases} \tag{M-2}$$

After solving the model (M-2) for the upper triangular part of the unknown q-ROFNs in the incomplete q-ROFPR A, the unknown q-ROFNs in the upper triangular parts of the incomplete q-ROFPR A can be derived. To obtain the numerical results of the model (M-2), the software package MATLAB is used in this paper. Based on **Definition 2.2**, we further derive those unknown q-ROFNs in the lower triangular part of the incomplete q-ROFPR A. Afterwards, the optimal objective value F^* and a complete q-ROFPR $A' = \left(\alpha'_{ij}\right)_{n\times n}$ are obtained. If $F^* \leqslant \delta$, where δ is a predefined consistency threshold value and $\delta \in [0,1]$, then the complete q-ROFPR A' is acceptable multiplicative consistent. Otherwise, this complete q-ROFPR A' is unacceptable consistent.

3.3. Models of deriving an acceptable multiplicative consistent q-ROFPR

Let $A = (\alpha_{ij})_{n\times n}$ be a q-ROFPR, where $\alpha_{ij} = \left(\mu_{ij},\nu_{ij}\right)$ is a q-ROFN, $i=1,2,\cdots,n, j=1,2,\cdots,n$ and $q \geqslant 1$. Let δ be the predefined consistency threshold value, where $\delta \in [0,1]$. According to **Definition 3.3**, it can be seen that if $MCI(A) \leqslant \delta$, then A is acceptable consistent. However, if we cannot ensure the acceptable multiplicative consistency of A, then the inequality "$MCI(A) \leqslant \delta$" may be not true. Generally speaking, q-ROFPRs are usually unacceptable multiplicative consistent following **Definition 3.1**. Therefore, we need to increase the consistency levels of q-ROFPRs. Besides, the adjustments should try to retain the original judgements of q-ROFPRs. Considering these two aspects, we propose the following optimization model:

$$f = \min\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(\left|\ln\left(\mu_{ij}\right)-\ln\left(\tilde{\mu}_{ij}\right)\right|+\left|\ln\left(\nu_{ij}\right)-\ln\left(\tilde{\nu}_{ij}\right)\right|\right)$$

$$s.t.\begin{cases} \sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n}\left|\ln\left(\tilde{\mu}_{ij}\right)+\ln\left(\tilde{\mu}_{jk}\right)+\ln\left(\tilde{\nu}_{ik}\right)-\ln\left(\tilde{\nu}_{ij}\right)-\ln\left(\tilde{\nu}_{jk}\right)-\ln\left(\tilde{\mu}_{ik}\right)\right| \\ \qquad \leqslant \frac{n(n-1)(n-2)\delta}{6}, \\ \tilde{\mu}_{ij},\tilde{\nu}_{ij} \in [0,1],\ i,j=1,2,\cdots,n,\ i<j, \\ \tilde{\mu}_{ij}^{q}+\tilde{\nu}_{ij}^{q} \leqslant 1,\ i,j=1,2,\cdots,n,\ i<j, \end{cases} \tag{M-3}$$

where A is an unacceptable multiplicative consistent q-ROFPR, $\tilde{A} = \left(\tilde{\alpha}_{ij}\right)_{n\times n}$ is the corresponding adjusted acceptable multiplicative consistent q-ROFPR, $\tilde{\alpha}_{ij} = \left(\tilde{\mu}_{ij},\tilde{\nu}_{ij}\right)$ is a q-ROFN, $i=1,2,\cdots,n, j=1,2,\cdots,n$ and $q \geqslant 1$.

The originality of the model (M-3) is that it not only derives an acceptable multiplicative consistent q-ROFPR from an unacceptable multiplicative consistent q-ROFPR, but also keeps the preference information in the original q-ROFPR as much as possible.

After removing the symbol "| |" of the absolute value in the model (M-3), the following model is derived:

$$f = \min\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(\phi_{ij}^{-}+\phi_{ij}^{+}+\varphi_{ij}^{-}+\varphi_{ij}^{+}\right)$$

$$s.t.\begin{cases} \ln\left(\mu_{ij}\right)-\ln\left(\tilde{\mu}_{ij}\right)+\phi_{ij}^{-}-\phi_{ij}^{+}=0,\ i,j=1,2,\cdots,n,\ i<j, \\ \ln\left(\nu_{ij}\right)-\ln\left(\tilde{\nu}_{ij}\right)+\varphi_{ij}^{-}-\varphi_{ij}^{+}=0,\ i,j=1,2,\cdots,n,\ i<j, \\ \sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n}\left(\psi_{ijk}^{-}+\psi_{ijk}^{+}\right) \leqslant \frac{n(n-1)(n-2)\delta}{6}, \\ \ln\left(\tilde{\mu}_{ij}\right)+\ln\left(\tilde{\mu}_{jk}\right)+\ln\left(\tilde{\nu}_{ik}\right)-\ln\left(\tilde{\nu}_{ij}\right)-\ln\left(\tilde{\nu}_{jk}\right)-\ln\left(\tilde{\mu}_{ik}\right)+\psi_{ijk}^{-}-\psi_{ijk}^{+}=0,\ i,j,k=1,2,\cdots,n,\ i<j<k, \\ \tilde{\mu}_{ij},\tilde{\nu}_{ij} \in [0,1],\ i,j=1,2,\cdots,n,\ i<j, \\ \tilde{\mu}_{ij}^{q}+\tilde{\nu}_{ij}^{q} \leqslant 1,\ i,j=1,2,\cdots,n,\ i<j, \\ \phi_{ij}^{-},\phi_{ij}^{+},\varphi_{ij}^{-},\varphi_{ij}^{+} \geqslant 0,\ i,j=1,2,\cdots,n,\ i<j, \\ \psi_{ijk}^{-},\psi_{ijk}^{+} \geqslant 0,\ i,j,k=1,2,\cdots,n,\ i<j<k. \end{cases} \tag{M-4}$$

The first constraint to the fourth constraint in the model (M-4) guarantee that the adjusted q-ROFPR $\widetilde{A}$ meets the requirement of acceptable consistency; the fifth constraint to the eighth constraint in the model (M-4) guarantee that the adjusted q-ROFPR $\widetilde{A}$ is still a q-ROFPR. To obtain the numerical results of the model (M-4), the software package MATLAB is used in this paper.

3.4. The consistency-derived priority weight vector

Definition 3.4. For a q-ROF priority weight vector $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$, where $\omega_i = \left(\mu_{\omega_i}, v_{\omega_i}\right)$ is a q-ROFN, $\mu_{\omega_i}, v_{\omega_i} \in [0,1]$, $\mu^q_{\omega_i} + v^q_{\omega_i} \leqslant 1$, $i = 1, 2, \cdots, n$ and $q \geqslant 1$, ω is called a normalized q-rung orthopair fuzzy priority weight vector if $\sum_{j=1, j\neq i}^{n} \mu^q_{\omega_j} \leqslant v^q_{\omega_i}$ and $\mu^q_{\omega_i} + n - 2 \geqslant \sum_{j=1, j\neq i}^{n} v^q_{\omega_j}$ for all $i = 1, 2, \cdots, n$ and $q \geqslant 1$.

Based on the normalized q-ROF priority vector $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$, where $\omega_i = \left(\mu_{\omega_i}, v_{\omega_i}\right)$ is a q-ROFN, $\mu_{\omega_i}, v_{\omega_i} \in [0,1]$, $\mu^q_{\omega_i} + v^q_{\omega_i} \leqslant 1$, $i = 1, 2, \cdots, n$ and $q \geqslant 1$, a multiplicative consistent q-ROFPR $B = (\beta_{ij})_{n\times n}$ can be established, where β_{ij} is a q-ROFN, shown as follows:

$$\beta_{ij} = \left(\mu_{\beta_{ij}}, v_{\beta_{ij}}\right) = \begin{cases} \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right), & \text{if} \quad i = j, \\ \left(\mu_{\omega_i} v_{\omega_j}, v_{\omega_i} \mu_{\omega_j}\right), & \text{if} \quad i \neq j, \end{cases} \tag{6}$$

for all $i, j = 1, 2, \cdots, n$.

Theorem 3.2. *Let $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ be a normalized q-ROF priority weight vector $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$, where $\omega_i = \left(\mu_{\omega_i}, v_{\omega_i}\right)$ is a q-ROFN, $\mu_{\omega_i}, v_{\omega_i} \in [0,1]$, $\mu^q_{\omega_i} + v^q_{\omega_i} \leqslant 1$, $i = 1, 2, \cdots, n$ and $q \geqslant 1$. The matrix $B = (\beta_{ij})_{n\times n}$ is a multiplicative consistent q-ROFPR, where β_{ij} is a q-ROFN and $\beta_{ij} = \left(\mu_{\beta_{ij}}, v_{\beta_{ij}}\right) = \begin{cases} \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right), & \text{if} \quad i = j, \\ \left(\mu_{\omega_i} v_{\omega_j}, v_{\omega_i} \mu_{\omega_j}\right), & \text{if} \quad i \neq j, \end{cases}$ for all $i, j = 1, 2, \cdots, n$.*

Proof. *For the matrix $B = (\beta_{ij})_{n\times n}$, where β_{ij} is a q-ROFN and $\beta_{ij} = \left(\mu_{\beta_{ij}}, v_{\beta_{ij}}\right) = \begin{cases} \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right), & \text{if} \quad i = j, \\ \left(\mu_{\omega_i} v_{\omega_j}, v_{\omega_i} \mu_{\omega_j}\right), & \text{if} \quad i \neq j, \end{cases}$ for all $i, j = 1, 2, \cdots, n$, we get $\mu_{\beta_{ii}} = \sqrt[q]{0.5}$, $v_{\beta_{ii}} = \sqrt[q]{0.5}$, $\mu_{\beta_{ij}} = \mu_{\omega_i} v_{\omega_j}$, $v_{\beta_{ij}} = v_{\omega_i} \mu_{\omega_j}$, $\mu^q_{\beta_{ij}} = \mu^q_{\omega_i} v^q_{\omega_j}$ and $v^q_{\beta_{ij}} = v^q_{\omega_i} \mu^q_{\omega_j}$.* Moreover, from the inequality $a^2 + b^2 \geqslant 2ab$ *for any two real numbers a and b, we have $\mu^q_{\beta_{ij}} = \mu^q_{\omega_i} v^q_{\omega_j} \leqslant \frac{\left(\mu^q_{\omega_i}\right)^2 + \left(v^q_{\omega_j}\right)^2}{2}$ and $v^q_{\beta_{ij}} = v^q_{\omega_i} \mu^q_{\omega_j} \leqslant \frac{\left(v^q_{\omega_i}\right)^2 + \left(\mu^q_{\omega_j}\right)^2}{2}$. Furthermore, because $\mu_{\omega_i}, v_{\omega_i} \in [0,1]$ for any $i = 1, 2, \cdots, n$ and $q \geqslant 1$, we can obtain $\mu^q_{\omega_i}, v^q_{\omega_j}, v^q_{\omega_i}, \mu^q_{\omega_j} \in [0,1]$. Therefore, we have $\left(\mu^q_{\omega_i}\right)^2 \leqslant \mu^q_{\omega_i}$, $\left(v^q_{\omega_j}\right)^2 \leqslant v^q_{\omega_j}$, $\left(v^q_{\omega_i}\right)^2 \leqslant v^q_{\omega_i}$ and $\left(\mu^q_{\omega_j}\right)^2 \leqslant \mu^q_{\omega_j}$. Then, because $\mu^q_{\omega_i} + v^q_{\omega_i} \leqslant 1$ for any $i = 1, 2, \cdots, n$, we have $\mu^q_{\omega_j} + v^q_{\omega_j} \leqslant 1$. Based on the above results, we obtain*

$$\begin{aligned} \mu^q_{\beta_{ij}} + v^q_{\beta_{ij}} = \mu^q_{\omega_i} v^q_{\omega_j} + v^q_{\omega_i} \mu^q_{\omega_j} &\leqslant \frac{\left(\mu^q_{\omega_i}\right)^2 + \left(v^q_{\omega_j}\right)^2 + \left(v^q_{\omega_i}\right)^2 + \left(\mu^q_{\omega_j}\right)^2}{2} \\ &\leqslant \frac{\mu^q_{\omega_i} + v^q_{\omega_j} + v^q_{\omega_i} + \mu^q_{\omega_j}}{2} = \frac{\left(\mu^q_{\omega_i} + v^q_{\omega_i}\right) + \left(\mu^q_{\omega_j} + v^q_{\omega_j}\right)}{2} \\ &\leqslant \frac{1+1}{2} = 1. \end{aligned}$$

In addition, from the matrix $B = (\beta_{ij})_{n\times n}$, where β_{ij} is a q-ROFN and $\beta_{ij} = \left(\mu_{\beta_{ij}}, v_{\beta_{ij}}\right) = \begin{cases} \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right), & \text{if} \quad i = j, \\ \left(\mu_{\omega_i} v_{\omega_j}, v_{\omega_i} \mu_{\omega_j}\right), & \text{if} \quad i \neq j, \end{cases}$ we have $\beta_{ji} = \left(\mu_{\beta_{ji}}, v_{\beta_{ji}}\right) = \begin{cases} \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right), & \text{if} \quad j = i, \\ \left(\mu_{\omega_j} v_{\omega_i}, v_{\omega_j} \mu_{\omega_i}\right), & \text{if} \quad j \neq i, \end{cases}$. Thus, we obtain $\mu_{\beta_{ij}} = \mu_{\omega_i} v_{\omega_j} = v_{\omega_j} \mu_{\omega_i} = v_{\beta_{ji}}$ and $v_{\beta_{ij}} = v_{\omega_i} \mu_{\omega_j} = \mu_{\omega_j} v_{\omega_i} = \mu_{\beta_{ji}}$. Therefore, based on **Definition 2.2**, we can see that the matrix $B = (\beta_{ij})_{n\times n}$ is a q-ROFPR. Furthermore, we have

$$\mu_{\beta_{ij}} \mu_{\beta_{jk}} v_{\beta_{ik}} = \mu_{\omega_i} v_{\omega_j} \mu_{\omega_j} v_{\omega_k} v_{\omega_i} \mu_{\omega_k} = v_{\omega_i} \mu_{\omega_j} v_{\omega_j} \mu_{\omega_k} \mu_{\omega_i} v_{\omega_k} = v_{\beta_{ij}} v_{\beta_{jk}} \mu_{\beta_{ik}}.$$

Based on **Definition 3.1**, it can be seen that the matrix B is multiplicatively consistent. **Q. E. D.**

Theorem 3.3. Let $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ be a normalized q-ROF priority vector, where $\omega_i = \left(\mu_{\omega_i}, \nu_{\omega_i}\right)$ is a q-ROFN, $\mu_{\omega_i}, \nu_{\omega_i} \in [0,1]$, $\mu_{\omega_i}^q + \nu_{\omega_i}^q \leqslant 1$, $\sum_{j=1, j\neq i}^n \mu_{\omega_j}^q \leqslant \nu_{\omega_i}^q$, $\mu_{\omega_i}^q + n - 2 \geqslant \sum_{j=1, j\neq i}^n \nu_{\omega_j}^q$, $i = 1, 2, \cdots, n$ and $q \geqslant 1$. Let $A = (\alpha_{ij})_{n\times n}$ be a q-ROFPR, where $\alpha_{ij} = \left(\mu_{ij}, \nu_{ij}\right)$ is a q-ROFN, $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, n$ and $q \geqslant 1$, shown as follows:

$$\alpha_{ij} = \left(\mu_{ij}, \nu_{ij}\right) = \begin{cases} \left(\sqrt[q]{0.5}, \sqrt[q]{0.5}\right), & \text{if } \ i = j, \\ \left(\mu_{\omega_i}\nu_{\omega_j}, \nu_{\omega_i}\mu_{\omega_j}\right), & \text{if } \ i < j, \\ \left(\nu_{\omega_j}\mu_{\omega_i}, \mu_{\omega_j}\nu_{\omega_i}\right), & \text{if } \ i > j, \end{cases} \tag{7}$$

Then, the q-ROFPR A given by Eq. (7) is multiplicatively consistent.

Based on Eq. (7), we get

$$\begin{cases} \ln\left(\mu_{ij}\right) = \ln\left(\mu_{\omega_i}\right) + \ln\left(\nu_{\omega_j}\right), \ i,j = 1, 2, \cdots, n, \ i < j, \\ \ln(\nu_{ij}) = \ln(\nu_{\omega_i}) + \ln\left(\mu_{\omega_j}\right), \ i,j = 1, 2, \cdots, n, \ i < j. \end{cases} \tag{8}$$

Generally speaking, Eq. (8) does not hold. Therefore, Eq. (8) is relaxed through using positive deviation variables, where

$$\begin{cases} \ln\left(\mu_{ij}\right) - \ln\left(\mu_{\omega_i}\right) - \ln\left(\nu_{\omega_j}\right) - \xi_{ij}^+ + \xi_{ij}^- = 0, \ i,j = 1, 2, \cdots, n, \ i < j, \\ \ln(\nu_{ij}) - \ln(\nu_{\omega_i}) - \ln\left(\mu_{\omega_j}\right) - \eta_{ij}^+ + \eta_{ij}^- = 0, \ i,j = 1, 2, \cdots, n, \ i < j. \end{cases} \tag{9}$$

To maximize the consistency of the q-ROFPR $A = (\alpha_{ij})_{n\times n}$, where $\alpha_{ij} = \left(\mu_{ij}, \nu_{ij}\right)$ is a q-ROFN, $\mu_{\omega_i}, \nu_{\omega_i} \in [0,1]$, and $\mu_{\omega_i}^q + \nu_{\omega_i}^q \leqslant 1$, we propose the following model to get the q-ROF priority weight vector $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$, where $\omega_i = \left(\mu_{\omega_i}, \nu_{\omega_i}\right)$ is a q-ROFN, $\mu_{\omega_i}, \nu_{\omega_i} \in [0,1]$, $\mu_{\omega_i}^q + \nu_{\omega_i}^q \leqslant 1$, i = 1, 2, ..., n and $q \geqslant 1$, shown as follows:

$$g = \min \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(\xi_{ij}^+ + \xi_{ij}^- + \eta_{ij}^+ + \eta_{ij}^-\right)$$

$$s.t. \begin{cases} \ln\left(\mu_{ij}\right) - \ln\left(\mu_{\omega_i}\right) - \ln\left(\nu_{\omega_j}\right) - \xi_{ij}^+ + \xi_{ij}^- = 0, \ i,j = 1, 2, \cdots, n, \ i < j, \\ \ln(\nu_{ij}) - \ln(\nu_{\omega_i}) - \ln\left(\mu_{\omega_j}\right) - \eta_{ij}^+ + \eta_{ij}^- = 0, \ i,j = 1, 2, \cdots, n, \ i < j, \\ \mu_{\omega_i}, \nu_{\omega_i} \in [0,1], \ i = 1, 2, \cdots, n, \\ \mu_{\omega_i}^q + \nu_{\omega_i}^q \leqslant 1, \ i = 1, 2, \cdots, n, \\ \sum_{j=1, j\neq i}^{n} \mu_{\omega_j}^q \leqslant \nu_{\omega_i}^q, \ i = 1, 2, \cdots, n, \\ \mu_{\omega_i}^q + n - 2 \geqslant \sum_{j=1, j\neq i}^{n} \nu_{\omega_j}^q, \ i = 1, 2, \cdots, n, \\ \xi_{ij}^+, \xi_{ij}^-, \eta_{ij}^+, \eta_{ij}^- \geqslant 0, \\ i,j = 1, 2, \cdots, n, \ i < j. \end{cases} \tag{M-5}$$

In the model (M-5), $A = (\alpha_{ij})_{n\times n}$ is a q-ROFPR, where $\alpha_{ij} = \left(\mu_{ij}, \nu_{ij}\right)$ is a q-ROFN, $\mu_{\omega_i}, \nu_{\omega_i} \in [0,1]$, $\mu_{\omega_i}^q + \nu_{\omega_i}^q \leqslant 1$, $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, n$ and $q \geqslant 1$; $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$ is the q-ROF priority weight vector, where $\omega_i = \left(\mu_{\omega_i}, \nu_{\omega_i}\right)$ is a q-ROFN, $\mu_{\omega_i}, \nu_{\omega_i} \in [0,1]$, $\mu_{\omega_i}^q + \nu_{\omega_i}^q \leqslant 1$, i = 1, 2, ..., n and $q \geqslant 1$; ξ_{ij}^+, ξ_{ij}^-, η_{ij}^+, and η_{ij}^- are positive deviation variables, where $i,j = 1, 2, \cdots, n$ and $i < j$. The former two constraints ensure that the obtained q-ROF priority weight vector ω can maximize the multiplicative consistency of the q-ROFPR $A = (\alpha_{ij})_{n\times n}$, where $\alpha_{ij} = \left(\mu_{ij}, \nu_{ij}\right)$ is a q-ROFN, $\mu_{\omega_i}, \nu_{\omega_i} \in [0,1]$, and $\mu_{\omega_i}^q + \nu_{\omega_i}^q \leqslant 1$, and the rest of constraints ensure that the obtained priority weight vector ω is a normalized q-ROF priority weight vector. To obtain the numerical results of the model (M-5), the software package MATLAB is used in this paper.

3.5. An individual decision making method with an incomplete q-ROFPR

For individual decision making with a q-ROFPR $A = (\alpha_{ij})_{n\times n}$, where $\alpha_{ij} = \left(\mu_{ij}, \nu_{ij}\right)$ is a q-ROFN, $\mu_{\omega_i}, \nu_{\omega_i} \in [0,1]$, $\mu_{\omega_i}^q + \nu_{\omega_i}^q \leqslant 1$, $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, n$ and $q \geqslant 1$, when the q-ROFPR A is acceptable multiplicative consistent, its q-

ROF priority vector is obtained. Otherwise, A should be adjusted by the model (M-4) until it is acceptable multiplicative consistent. In the following, we propose an individual decision making method with an incomplete q-ROFPR, called **Algorithm 1**. Let δ be a predefined consistency threshold value, where $\delta \in [0,1]$. The proposed **Algorithm 1** for individual decision making with a q-ROFPR $A = (\alpha_{ij})_{n\times n}$ is now presented as follows:

Algorithm 1:

Step 1: If the q-ROFPR A is complete, then go to **Step 2**. Otherwise, substitute the incomplete q-ROFPR A into the model (M-2) to determine the unknown judgements in the q-ROFPR A by solving this model, which is still denoted by A.

Step 2: Based on Eq. (4), calculate the consistency index $MCI(A)$ of the q-ROFPR A.

Step 3: According to **Definition 3.3,** judge whether the q-ROFPR A is acceptable consistent.

(i) If $MCI(A) \leqslant \delta$, where $\delta \in [0,1]$, then let the acceptable multiplicative consistent q-ROFPR $\widetilde{A} = A$ and go to **Step 4**.

(ii) If $MCI(A) > \delta$, where $\delta \in [0,1]$, then apply the model (M-4) to get an acceptable multiplicative consistent q-ROFPR $\widetilde{A}$ corresponding to A.

Step 4: Substitute the obtained acceptable multiplicative consistent q-ROFPR $\widetilde{A}$ into the model (M-5) to obtain the q-ROF priority weight vector $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T$, where $\omega_i = \left(\mu_{\omega_i}, v_{\omega_i}\right)$ is a q-ROFN, $\mu_{\omega_i}, v_{\omega_i} \in [0,1]$, $\mu^q_{\omega_i} + v^q_{\omega_i} \leqslant 1$, $i = 1,2,\cdots,n$ and $q \geqslant 1$.

Step 5: Based on **Definition 2.1**, compute $S(\omega_i)$ and $H(\omega_i)$ for ω_i, respectively, where $\omega_i = \left(\mu_{\omega_i}, v_{\omega_i}\right)$, $S(\omega_i) = \mu_{\omega_i} - v_{\omega_i}$, $H(\omega_i) = \mu_{\omega_i} + v_{\omega_i}$ and $i = 1,2,\cdots,n$. For any $i,j = 1,2,\cdots,n$ and $i \neq j$, if $S(\omega_i) > S(\omega_j)$, then alternative x_i is better than alternative x_j; if $S(\omega_i) = S(\omega_j)$ and $H(\omega_i) > H(\omega_j)$, then alternative x_i is better than alternative x_j; if $S(\omega_i) = S(\omega_j)$ and $H(\omega_i) = H(\omega_j)$, then alternative x_i and alternative x_j have the same ranking order.

In the following, we show the utilization of the proposed **Algorithm 1** for individual decision making.

Example 3.1. Let $X = \{x_1, x_2, x_3, x_4\}$ be a set of alternatives and let $A = (\alpha_{ij})_{4\times 4}$ be an incomplete q-ROFPR on X given by the DM, where $\alpha_{ij} = \left(\mu_{ij}, v_{ij}\right)$ is a q-ROFN, $i = 1,2,3,4$, $j = 1,2,3,4$ and $q \geqslant 1$, shown as follows:

$$A = \begin{pmatrix} \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.7, 0.2) & (-,-) & (0.5, 0.9) \\ (0.2, 0.7) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.7, 0.1) & (-, 0.5) \\ (-,-) & (0.1, 0.7) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.8, 0.1) \\ (0.9, 0.5) & (0.5, -) & (0.1, 0.8) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) \end{pmatrix}$$

where the symbol "$-$" denotes an unknown value. Let the consistency threshold value $\delta = 0.1$ and let $q = 3$. We use the proposed "**Algorithm 1**" to deal with this individual decision making problem, shown as follows:

[Step 1] Complete the missing values in the incomplete q-ROFPR $A = (\alpha_{ij})_{4\times 4}$ according to the model (M-2) to get the complete q-ROFPR $A = (\alpha_{ij})_{4\times 4}$, where $i = 1,2,3,4$, $j = 1,2,3,4$ and $q \geqslant 1$, shown as follows:

$$A = \begin{pmatrix} \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.7, 0.2) & (0.9092, 0.5501) & (0.5, 0.9) \\ (0.2, 0.7) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.7, 0.1) & (0.9092, 0.5) \\ (0.5501, 0.9092) & (0.1, 0.7) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.8, 0.1) \\ (0.9, 0.5) & (0.5, 0.9092) & (0.1, 0.8) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) \end{pmatrix}$$

[Step 2] Based on Eq. (4), we get $MCI(A) = 2.933$.

[Step 3] Because $MCI(A) > \delta$, where $MCI(A) = 2.933$ and the consistency threshold value $\delta = 0.1$, it can be seen that the complete q-ROFPR $A = (\alpha_{ij})_{4\times 4}$ is unacceptable consistent. Then, we derive the acceptable consistent q-ROFPR $\widetilde{A}$ corresponding to the q-ROFPR A, shown as follows:

$$\widetilde{A} = \begin{pmatrix} \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.6969, 0.8012) & (0.9092, 0.5501) & (0.9026, 0.6019) \\ (0.8012, 0.6969) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.6975, 0.3663) & (0.9092, 0.5) \\ (0.5501, 0.9092) & (0.3663, 0.6975) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.7855, 0.7103) \\ (0.6019, 0.9026) & (0.5, 0.9092) & (0.7103, 0.7855) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) \end{pmatrix}$$

[Step 4] The optimal q-ROF priority weight vector $\omega = (\omega_1, \omega_2, \omega_3, \omega_4)^T$ is obtained from the acceptable multiplicative consistent q-ROFPR $\widetilde{A}$ obtained in **Step 3**, where $\omega_1 = (0.6654, 0.8902)$, $\omega_2 = (0.7272, 0.8506)$, $\omega_3 = (0.4901, 0.9591)$ and $\omega_4 = (0.5878, 0.9271)$.

[Step 5] Based on the score function S presented in **Definition 2.1**, we get the score values $S(\omega_1)$, $S(\omega_2)$, $S(\omega_3)$ and $S(\omega_4)$ of the q-ROF priority weights ω_1, ω_2, ω_3 and ω_4, respectively, where $\omega_1 = (0.6654, 0.8902)$, $\omega_2 = (0.7272, 0.8506)$, $\omega_3 = (0.4901, 0.9591)$, $\omega_4 = (0.5878, 0.9271)$, $S(\omega_1) = -0.4108$, $S(\omega_2) = -0.2308$, $S(\omega_3) = -0.7646$ and $S(\omega_4) = -0.5938$. Because $S(\omega_2) > S(\omega_1) > S(\omega_4) > S(\omega_3)$, where $S(\omega_1) = -0.4108$, $S(\omega_2) = -0.2308$, $S(\omega_3) = -0.7646$ and $S(\omega_4) = -0.5938$, the ranking order of the alternatives is "$x_2 \succ x_1 \succ x_4 \succ x_3$".

By comparing the proposed **Algorithm 1** with Zhang et al.'s method [47] for individual decision making, we can see that the proposed **Algorithm 1** has the capability for individual decision making with incomplete q-ROFPRs, whereas Zhang *et al.*'s method [47] does not have the capability for individual decision making with incomplete q-ROFPRs. Thus, Zhang *et al.*'s method [47] cannot deal with **Example 3.1** shown above.

4. A new approach of GDM with incomplete *q*-ROFPRs

Let $X = \{x_1, x_2, \cdots, x_n\}$ be a set of alternatives and let $D = \{d_1, d_2, \cdots, d_m\}$ be a set of DMs to evaluate the alternatives. Let $A^s = \left(\alpha_{ij}^s\right)_{n\times n}$ be a q-ROFPR given by DM d_s for alternative x_i over alternative x_j, where $\alpha_{ij}^s = \left(\mu_{ij}^s, \nu_{ij}^s\right)$ is a q-ROFN, $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, n$, $s = 1, 2, \cdots, m$ and $q \geqslant 1$. In the following, we propose a consensus index of q-ROFPRs and propose an approach for increasing the consensus level of q-ROFPRs.

4.1. A consensus index of q-ROFPRs

In the framework of GDM, the consensus measure and the consensus checking of q-ROFPRs should be implemented. This section begins with the development of the consensus measure of q-ROFPRs in GDM.

Lemma 4.1 [30,34]. If $z_s > 0$, $w_s > 0$, $s = 1, 2, \cdots, m$ and $\sum_{s=1}^m w_s = 1$, then $\prod_{s=1}^m z_s^{w_s} \leqslant \sum_{s=1}^m w_s z_s$, with the equality hold if and only if $z_1 = z_2 = \cdots = z_m$.

Theorem 4.1. Let $A^s = \left(\alpha_{ij}^s\right)_{n\times n}$ be a q-ROFPR given by DM d_s for alternative x_i over alternative x_j, where $\alpha_{ij}^s = \left(\mu_{ij}^s, \nu_{ij}^s\right)$ is a q-ROFN, $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, n$, $s = 1, 2, \cdots, m$ and $q \geqslant 1$. Let $w = (w_1, w_2, \cdots, w_m)^T$ be the weight vector, where $w_s \in [0, 1]$ denotes the weight of DM d_s, $s = 1, 2, \cdots, m$ and $\sum_{s=1}^m w_s = 1$. Let the matrix $A^c = \left(\alpha_{ij}^c\right)_{n\times n}$, where $\alpha_{ij}^c = \left(\mu_{ij}^c, \nu_{ij}^c\right) = \left(\prod_{s=1}^m \left(\mu_{ij}^s\right)^{w_s}, \prod_{s=1}^m \left(\nu_{ij}^s\right)^{w_s}\right)$, then A^c is q-ROFPR, which is called the collective q-ROFPR.

Proof.

For any $i, j = 1, 2, \cdots, n$, we obtain

$$
\begin{cases}
0 \leqslant \mu_{ij}^c = \prod\limits_{s=1}^m \left(\mu_{ij}^s\right)^{w_s} \leqslant \prod\limits_{s=1}^m 1^{w_s} = 1,\ 0 \leqslant \nu_{ij}^c = \prod\limits_{s=1}^m \left(\nu_{ij}^s\right)^{w_s} \leqslant \prod\limits_{s=1}^m 1^{w_s} = 1, \\
\mu_{ii}^c = \prod\limits_{s=1}^m \left(\mu_{ii}^s\right)^{w_s} = \prod\limits_{s=1}^m \left(\sqrt[q]{0.5}\right)^{w_s} = \left(\sqrt[q]{0.5}\right)^{\sum_{s=1}^m w_s} = \sqrt[q]{0.5}, \\
\nu_{ii}^c = \prod\limits_{s=1}^m \left(\nu_{ii}^s\right)^{w_s} = \prod\limits_{s=1}^m \left(\sqrt[q]{0.5}\right)^{w_s} = \left(\sqrt[q]{0.5}\right)^{\sum_{s=1}^m w_s} = \sqrt[q]{0.5}, \\
\mu_{ij}^c = \prod\limits_{s=1}^m \left(\mu_{ij}^s\right)^{w_s} = \prod\limits_{s=1}^m \left(\nu_{ji}^s\right)^{w_s} = \nu_{ji}^c,\ \nu_{ij}^c = \prod\limits_{s=1}^m \left(\nu_{ij}^s\right)^{w_s} = \prod\limits_{s=1}^m \left(\mu_{ji}^s\right)^{w_s} = \mu_{ji}^c, \\
\left(\mu_{ij}^c\right)^q + \left(\nu_{ij}^c\right)^q = \left(\prod\limits_{s=1}^m \left(\mu_{ij}^s\right)^{w_s}\right)^q + \left(\prod\limits_{s=1}^m \left(\nu_{ij}^s\right)^{w_s}\right)^q = \prod\limits_{s=1}^m \left(\left(\mu_{ij}^s\right)^q\right)^{w_s} + \prod\limits_{s=1}^m \left(\left(\nu_{ij}^s\right)^q\right)^{w_s} \\
\leqslant \sum\limits_{s=1}^m \left(w_s \left(\mu_{ij}^s\right)^q\right) + \sum\limits_{s=1}^m \left(w_s \left(\nu_{ij}^s\right)^q\right) = \sum\limits_{s=1}^m \left(w_s \left(\left(\mu_{ij}^s\right)^q + \left(\nu_{ij}^s\right)^q\right)\right) \leqslant \sum\limits_{s=1}^m w_s = 1.
\end{cases}
$$

Based on **Definition 2.2**, the matrix A^c is a q-ROFPR, which is called the collective q-ROFPR. **Q. E. D.**

Theorem 4.2. *The collective q-ROFPR* $A^c = \left(\alpha_{ij}^c\right)_{n\times n}$ *is acceptable consistent, where* $\alpha_{ij}^c = \left(\mu_{ij}^c, \nu_{ij}^c\right) = \left(\prod_{s=1}^m \left(\mu_{ij}^s\right)^{w_s}, \prod_{s=1}^m \left(\nu_{ij}^s\right)^{w_s}\right)$ *is a q-ROFN, if all q-ROFPRs* $A^s = \left(\alpha_{ij}^s\right)_{n\times n}$ $(s = 1, 2, \cdots, m)$ *are acceptable consistent, where* $\alpha_{ij}^s = \left(\mu_{ij}^s, \nu_{ij}^s\right)$ *is a q-ROFN,* $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, n$, $s = 1, 2, \cdots, m$ *and* $q \geqslant 1$.

Proof. Following the acceptable consistency of A^s and based on **Definition 3.3**, we obtain

$$\begin{aligned}
MCI(A^c) &= \frac{6\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n}\left|\ln\left(\mu_{ij}^c\right)+\ln\left(\mu_{jk}^c\right)+\ln\left(v_{ik}^c\right)-\ln\left(v_{ij}^c\right)-\ln\left(v_{jk}^c\right)-\ln\left(\mu_{ik}^c\right)\right|}{n(n-1)(n-2)} \\
&= \frac{6\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n}\left|\ln\left(\prod_{s=1}^m\left(\mu_{ij}^s\right)^{w_s}\right)+\ln\left(\prod_{s=1}^m\left(\mu_{jk}^s\right)^{w_s}\right)+\ln\left(\prod_{s=1}^m\left(v_{ik}^s\right)^{w_s}\right)-\ln\left(\prod_{s=1}^m\left(v_{ij}^s\right)^{w_s}\right)-\ln\left(\prod_{s=1}^m\left(v_{jk}^s\right)^{w_s}\right)-\ln\left(\prod_{s=1}^m\left(\mu_{ik}^s\right)^{w_s}\right)\right|}{n(n-1)(n-2)} \\
&= \frac{6\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n}\left|\sum_{s=1}^m\left(w_s\ln\left(\mu_{ij}^s\right)\right)+\sum_{s=1}^m\left(w_s\ln\left(\mu_{jk}^s\right)\right)+\sum_{s=1}^m\left(w_s\ln\left(v_{ik}^s\right)\right)-\sum_{s=1}^m\left(w_s\ln\left(v_{ij}^s\right)\right)-\sum_{s=1}^m\left(w_s\ln\left(v_{jk}^s\right)\right)-\sum_{s=1}^m\left(w_s\ln\left(\mu_{ik}^s\right)\right)\right|}{n(n-1)(n-2)} \\
&= \frac{6\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n}\left|\sum_{s=1}^m\left(w_s\left(\ln\left(\mu_{ij}^s\right)+\ln\left(\mu_{jk}^s\right)+\ln\left(v_{ik}^s\right)-\ln\left(v_{ij}^s\right)-\ln\left(v_{jk}^s\right)-\ln\left(\mu_{ik}^s\right)\right)\right)\right|}{n(n-1)(n-2)} \\
&\leqslant \frac{6\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n}\sum_{s=1}^m\left(w_s\left|\ln\left(\mu_{ij}^s\right)+\ln\left(\mu_{jk}^s\right)+\ln\left(v_{ik}^s\right)-\ln\left(v_{ij}^s\right)-\ln\left(v_{jk}^s\right)-\ln\left(\mu_{ik}^s\right)\right|\right)}{n(n-1)(n-2)} \\
&= \sum_{s=1}^m\left(w_s\left(\frac{6\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{k=j+1}^{n}\left|\ln\left(\mu_{ij}^s\right)+\ln\left(\mu_{jk}^s\right)+\ln\left(v_{ik}^s\right)-\ln\left(v_{ij}^s\right)-\ln\left(v_{jk}^s\right)-\ln\left(\mu_{ik}^s\right)\right|}{n(n-1)(n-2)}\right)\right) \\
&= \sum_{s=1}^m\left(w_s MCI(A^s)\right) \leqslant \sum_{s=1}^m\left(w_s\delta\right) = \delta,
\end{aligned}$$

where δ is a consistency threshold value and $\delta \in [0,1]$. Thus, A^c is acceptable multiplicatively consistent. **Q. E. D.**

Definition 4.1. Let $A^s = \left(\alpha_{ij}^s\right)_{n\times n}$ and $A^t = \left(\alpha_{ij}^t\right)_{n\times n}$ be any two q-ROFPRs, where $\alpha_{ij}^s = \left(\mu_{ij}^s, v_{ij}^s\right)$ is a q-ROFN, $\alpha_{ij}^t = \left(\mu_{ij}^t, v_{ij}^t\right)$ is a q-ROFN, $i = 1,2,\cdots,n$, $j = 1,2,\cdots,n$ and $q \geqslant 1$. The correlation coefficient $CC(A^s, A^t)$ between the q-ROFPRs $A^s = \left(\alpha_{ij}^s\right)_{n\times n}$ and $A^t = \left(\alpha_{ij}^t\right)_{n\times n}$ is defined as follows:

$$CC(A^s,A^t) = \frac{1}{2}\left(\frac{\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(\mu_{ij}^s\mu_{ij}^t\right)}{\sqrt{\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(\mu_{ij}^s\right)^2}\sqrt{\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(\mu_{ij}^t\right)^2}} + \frac{\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(v_{ij}^s v_{ij}^t\right)}{\sqrt{\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(v_{ij}^s\right)^2}\sqrt{\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(v_{ij}^t\right)^2}}\right) \tag{10}$$

where $0 < CC(A^s,A^t) \leqslant 1$. If $A^s = A^t$, then $CC(A^s,A^t) = 1$.

Based on **Definition 4.1**, we present a formula to calculate the weight w_s of DM d_s, shown as follows:

$$w_s = \frac{\sum_{t=1,2,\cdots,m,\ t\neq s} CC(A^s,A^t)}{\sum_{s=1,2,\cdots,m}\sum_{t=1,2,\cdots,m,\ t\neq s} CC(A^s,A^t)} \tag{11}$$

where $s = 1,2,\cdots,m$.

Definition 4.2. Let $A^s = \left(\alpha_{ij}^s\right)_{n\times n}$ be a q-ROFPR, where $\alpha_{ij}^s = \left(\mu_{ij}^s, v_{ij}^s\right)$ is a q-ROFN, $i = 1,2,\cdots,n$, $j = 1,2,\cdots,n$, $s = 1,2,\cdots,m$ and $q \geqslant 1$, and let $A^c = \left(\alpha_{ij}^c\right)_{n\times n}$ be a collective q-ROFPR, where $\alpha_{ij}^c = \left(\mu_{ij}^c, v_{ij}^c\right) = \left(\prod_{s=1}^m\left(\mu_{ij}^s\right)^{w_s}, \prod_{s=1}^m\left(v_{ij}^s\right)^{w_s}\right)$ is a q-ROFPR. The consensus index $GCI(A^s)$ of A^s is defined as follows:

$$GCI(A^s) = \frac{1}{n(n-1)}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(\left|\ln\left(\mu_{ij}^s\right) - \ln\left(\mu_{ij}^c\right)\right| + \left|\ln\left(v_{ij}^s\right) - \ln\left(v_{ij}^c\right)\right|\right) \tag{12}$$

where $i = 1,2,\cdots,n$, $j = 1,2,\cdots,n$, $s = 1,2,\cdots,m$ and $q \geqslant 1$.

Let τ be a consensus threshold value, where $\tau \in [0,1]$. If $GCI(A^s) \leqslant \tau$ for all $s = 1,2,\cdots,m$, then each individual q-ROFPR $A^s = \left(\alpha_{ij}^s\right)_{n\times n}$ satisfy the consensus requirement. Otherwise, we need to improve the consensus levels of some q-ROFPRs. Let $GCI(A^s) = \max_{1\leqslant t\leqslant m}\left\{GCI(A^t)\right\} > \tau$, where $\tau \in [0,1]$.

Theorem 4.3. *Let* $A^s = \left(\alpha_{ij}^s\right)_{n\times n}$ *be a q-ROFPR, where* $\alpha_{ij}^s = \left(\mu_{ij}^s, v_{ij}^s\right)$ *is a q-ROFN,* $i = 1,2,\cdots,n$, $j = 1,2,\cdots,n$, $s = 1,2,\cdots,m$ *and* $q \geqslant 1$, *and let* $A^c = \left(\alpha_{ij}^c\right)_{n\times n}$ *be a collective q-ROFPR, where* $\alpha_{ij}^c = \left(\mu_{ij}^c, v_{ij}^c\right) = \left(\prod_{s=1}^m\left(\mu_{ij}^s\right)^{w_s}, \prod_{s=1}^m\left(v_{ij}^s\right)^{w_s}\right)$ *is a q-ROFPR,* $i = 1,2,\cdots,n$, $j = 1,2,\cdots,n$ *and* $q \geqslant 1$. *Let* $A'^s = \left(\alpha'^s_{ij}\right)_{n\times n}$ *be the adjusted q-ROFPR, where* $\alpha'^s_{ij} = \left(\mu'^s_{ij}, v'^s_{ij}\right) = \left(\left(\mu_{ij}^s\right)^{\lambda_{ij}^s} \times \left(\mu_{ij}^c\right)^{1-\lambda_{ij}^s}, \left(v_{ij}^s\right)^{\theta_{ij}^s} \times \left(v_{ij}^c\right)^{1-\theta_{ij}^s}\right)$, *and* $\lambda_{ij}^s, \theta_{ij}^s \in [0,1]$. *Then,* $GCI\left(A'^s\right) \leqslant GCI(A^s)$, *where* $GCI\left(A'^s\right) = \frac{1}{n(n-1)}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left(\left|\ln\left(\mu'^s_{ij}\right) - \ln\left(\mu'^c_{ij}\right)\right| + \left|\ln\left(v'^s_{ij}\right) - \ln\left(v'^c_{ij}\right)\right|\right)$, $A'^c = \left(\alpha'^c_{ij}\right)_{n\times n}$, $\alpha'^c_{ij} = \left(\mu'^c_{ij}, v'^c_{ij}\right) = \left(\left(\mu'^s_{ij}\right)^{w_s} \times \prod_{t=1,t\neq s}^m\left(\mu_{ij}^t\right)^{w_t}, \left(v'^s_{ij}\right)^{w_s} \times \prod_{t=1,t\neq s}^m\left(v_{ij}^t\right)^{w_t}\right)$ *is a q-ROFPR,* $i=1,2,\cdots,n$, $j=1,2,\cdots,n$ *and* $q \geqslant 1$.

Proof. Firstly, we prove

$$\left|\ln\left(\mu'^{s}_{ij}\right)-\ln\left(\mu'^{c}_{ij}\right)\right|+\left|\ln\left(v'^{s}_{ij}\right)-\ln\left(v'^{c}_{ij}\right)\right|\leqslant\left|\ln\left(\mu^{s}_{ij}\right)-\ln\left(\mu^{c}_{ij}\right)\right|+\left|\ln\left(v^{s}_{ij}\right)-\ln\left(v^{c}_{ij}\right)\right|. \tag{13}$$

Because $A'^{c}=\left(\alpha'^{c}_{ij}\right)_{n\times n}$, where $\alpha'^{c}_{ij}=\left(\mu'^{c}_{ij},v'^{c}_{ij}\right)=\left(\left(\mu'^{s}_{ij}\right)^{w_s}\times\prod_{t=1,t\neq s}^{m}\left(\mu^{t}_{ij}\right)^{w_t},\left(v'^{s}_{ij}\right)^{w_s}\times\prod_{t=1,t\neq s}^{m}\left(v^{t}_{ij}\right)^{w_t}\right)$ is a q-ROFPR, we derive

$$\begin{aligned}\ln\left(\mu'^{c}_{ij}\right)&=\ln\left(\left(\mu'^{s}_{ij}\right)^{w_s}\times\prod_{t=1,t\neq s}^{m}\left(\mu^{t}_{ij}\right)^{w_t}\right)=w_s\ln\left(\mu'^{s}_{ij}\right)+\sum_{t=1,t\neq s}^{m}w_t\ln\left(\mu^{t}_{ij}\right)\\&=w_s\ln\left(\left(\mu^{s}_{ij}\right)^{\lambda^{s}_{ij}}\times\left(\mu^{c}_{ij}\right)^{1-\lambda^{s}_{ij}}\right)+\sum_{t=1,t\neq s}^{m}w_t\ln\left(\mu^{t}_{ij}\right)\\&=w_s\left(\lambda^{s}_{ij}\ln\left(\mu^{s}_{ij}\right)+\left(1-\lambda^{s}_{ij}\right)\ln\left(\mu^{c}_{ij}\right)\right)+\sum_{t=1}^{m}w_t\ln\left(\mu^{t}_{ij}\right)-w_s\ln\left(\mu^{s}_{ij}\right)\\&=\ln\left(\mu^{c}_{ij}\right)+w_s\left(1-\lambda^{s}_{ij}\right)\ln\left(\mu^{c}_{ij}\right)-w_s\left(1-\lambda^{s}_{ij}\right)\ln\left(\mu^{s}_{ij}\right)\end{aligned} \tag{14}$$

and

$$\begin{aligned}\ln\left(v'^{c}_{ij}\right)&=\ln\left(\left(v'^{s}_{ij}\right)^{w_s}\times\prod_{t=1,t\neq s}^{m}\left(v^{t}_{ij}\right)^{w_t}\right)=w_s\ln\left(v'^{s}_{ij}\right)+\sum_{t=1,t\neq s}^{m}w_t\ln\left(v^{t}_{ij}\right)\\&=w_s\ln\left(\left(v^{s}_{ij}\right)^{\theta^{s}_{ij}}\times\left(v^{c}_{ij}\right)^{1-\theta^{s}_{ij}}\right)+\sum_{t=1,t\neq s}^{m}w_t\ln\left(v^{t}_{ij}\right)\\&=w_s\left(\theta^{s}_{ij}\ln\left(v^{s}_{ij}\right)+\left(1-\theta^{s}_{ij}\right)\ln\left(v^{c}_{ij}\right)\right)+\sum_{t=1}^{m}w_t\ln\left(v^{t}_{ij}\right)-w_s\ln\left(v^{s}_{ij}\right)\\&=\ln\left(v^{c}_{ij}\right)+w_s\left(1-\theta^{s}_{ij}\right)\ln\left(v^{c}_{ij}\right)-w_s\left(1-\theta^{s}_{ij}\right)\ln\left(v^{s}_{ij}\right).\end{aligned} \tag{15}$$

Based on Eq. (14), we obtain

$$\begin{aligned}\left|\ln\left(\mu'^{s}_{ij}\right)-\ln\left(\mu'^{c}_{ij}\right)\right|&=\left|\lambda^{s}_{ij}\ln\left(\mu^{s}_{ij}\right)+\left(1-\lambda^{s}_{ij}\right)\ln\left(\mu^{c}_{ij}\right)-\left(\ln\left(\mu^{c}_{ij}\right)+w_s\left(1-\lambda^{s}_{ij}\right)\ln\left(\mu^{c}_{ij}\right)-w_s\left(1-\lambda^{s}_{ij}\right)\ln\left(\mu^{s}_{ij}\right)\right)\right|\\&=\left|\lambda^{s}_{ij}\ln\left(\mu^{s}_{ij}\right)+\left(1-\lambda^{s}_{ij}\right)\ln\left(\mu^{c}_{ij}\right)-\ln\left(\mu^{c}_{ij}\right)-w_s\left(1-\lambda^{s}_{ij}\right)\ln\left(\mu^{c}_{ij}\right)+w_s\left(1-\lambda^{s}_{ij}\right)\ln\left(\mu^{s}_{ij}\right)\right|\\&=\left|\left(\lambda^{s}_{ij}+w_s\left(1-\lambda^{s}_{ij}\right)\right)\ln\left(\mu^{s}_{ij}\right)-\left(\lambda^{s}_{ij}+w_s\left(1-\lambda^{s}_{ij}\right)\right)\ln\left(\mu^{c}_{ij}\right)\right|\\&=\left(\lambda^{s}_{ij}+w_s\left(1-\lambda^{s}_{ij}\right)\right)\left|\ln\left(\mu^{s}_{ij}\right)-\ln\left(\mu^{c}_{ij}\right)\right|\\&=\left((1-w_s)\lambda^{s}_{ij}+w_s\right)\left|\ln\left(\mu^{s}_{ij}\right)-\ln\left(\mu^{c}_{ij}\right)\right|.\end{aligned} \tag{16}$$

Similarly, based on Eq. (15), we derive

$$\begin{aligned}\left|\ln\left(v'^{s}_{ij}\right)-\ln\left(v'^{c}_{ij}\right)\right|&=\left|\theta^{s}_{ij}\ln\left(v^{s}_{ij}\right)+\left(1-\theta^{s}_{ij}\right)\ln\left(v^{c}_{ij}\right)-\ln\left(v^{c}_{ij}\right)+w_s\left(1-\theta^{s}_{ij}\right)\ln\left(v^{c}_{ij}\right)-w_s\left(1-\theta^{s}_{ij}\right)\ln\left(v^{s}_{ij}\right)\right|\\&=\left|\theta^{s}_{ij}\ln\left(v^{s}_{ij}\right)+\left(1-\theta^{s}_{ij}\right)\ln\left(v^{c}_{ij}\right)-\ln\left(v^{c}_{ij}\right)-w_s\left(1-\theta^{s}_{ij}\right)\ln\left(v^{c}_{ij}\right)+w_s\left(1-\theta^{s}_{ij}\right)\ln\left(v^{s}_{ij}\right)\right|\\&=\left|\left(\theta^{s}_{ij}+w_s\left(1-\theta^{s}_{ij}\right)\right)\ln\left(v^{s}_{ij}\right)-\theta^{s}_{ij}\ln\left(v^{c}_{ij}\right)-w_s\left(1-\theta^{s}_{ij}\right)\ln\left(v^{c}_{ij}\right)\right|\\&=\left|\left(\theta^{s}_{ij}+w_s\left(1-\theta^{s}_{ij}\right)\right)\ln\left(v^{s}_{ij}\right)-\left(\theta^{s}_{ij}+w_s\left(1-\theta^{s}_{ij}\right)\right)\ln\left(v^{c}_{ij}\right)\right|\\&=\left(\theta^{s}_{ij}+w_s\left(1-\theta^{s}_{ij}\right)\right)\left|\ln\left(v^{s}_{ij}\right)-\ln\left(v^{c}_{ij}\right)\right|\\&=\left((1-w_s)\theta^{s}_{ij}+w_s\right)\left|\ln\left(v^{s}_{ij}\right)-\ln\left(v^{c}_{ij}\right)\right|.\end{aligned} \tag{17}$$

Based on Eqs. (16) and (17), we get

$$\begin{aligned}&\left|\ln\left(\mu'^{s}_{ij}\right)-\ln\left(\mu'^{c}_{ij}\right)\right|+\left|\ln\left(v'^{s}_{ij}\right)-\ln\left(v'^{c}_{ij}\right)\right|\\&=\left((1-w_s)\lambda^{s}_{ij}+w_s\right)\left|\ln\left(\mu^{s}_{ij}\right)-\ln\left(\mu^{c}_{ij}\right)\right|+\left((1-w_s)\theta^{s}_{ij}+w_s\right)\left|\ln\left(v^{s}_{ij}\right)-\ln\left(v^{c}_{ij}\right)\right|\\&\leqslant((1-w_s)+w_s)\left|\ln\left(\mu^{s}_{ij}\right)-\ln\left(\mu^{c}_{ij}\right)\right|+((1-w_s)+w_s)\left|\ln\left(v^{s}_{ij}\right)-\ln\left(v^{c}_{ij}\right)\right|\\&=\left|\ln\left(\mu^{s}_{ij}\right)-\ln\left(\mu^{c}_{ij}\right)\right|+\left|\ln\left(v^{s}_{ij}\right)-\ln\left(v^{c}_{ij}\right)\right|.\end{aligned}$$

Then, based on Eq. (12), we get $GCI\left(A'^{s}\right) \leqslant GCI(A^{s})$. **Q. E. D.**

It should be noted that **Theorem 4.3** does not consider the consistency of the adjusted q-ROFPRs. To guarantee that the consistency does not change, we establish a model to revise the consensus for each individual q-ROFPR:

$$\varepsilon = \max \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(\lambda_{ij}^{s} + \theta_{ij}^{s}\right)$$

$$\begin{cases} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(\left((1-w_s)\lambda_{ij}^{s} + w_s\right) \left| \ln\left(\mu_{ij}^{s}\right) - \ln\left(\mu_{ij}^{c}\right) \right| + \left((1-w_s)\theta_{ij}^{s} + w_s\right) \left| \ln\left(v_{ij}^{s}\right) - \ln\left(v_{ij}^{c}\right) \right| \right) \\ \qquad \leqslant n(n-1)\tau, \\ \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} \left| \lambda_{ij}^{s}\ln\left(\mu_{ij}^{s}\right) + \left(1-\lambda_{ij}^{s}\right)\ln\left(\mu_{ij}^{c}\right) + \lambda_{jk}^{s}\ln\left(\mu_{jk}^{s}\right) + \left(1-\lambda_{jk}^{s}\right)\ln\left(\mu_{jk}^{c}\right) + \right. \\ \theta_{ik}^{s}\ln(v_{ik}^{s}) + (1-\theta_{ik}^{s})\ln(v_{ik}^{c}) - \theta_{ij}^{s}\ln\left(v_{ij}^{s}\right) - \left(1-\theta_{ij}^{s}\right)\ln\left(v_{ij}^{c}\right) - \theta_{jk}^{s}\ln\left(v_{jk}^{s}\right) - \\ \left. \left(1-\theta_{jk}^{s}\right)\ln\left(v_{jk}^{c}\right) - \lambda_{ik}^{s}\ln(\mu_{ik}^{s}) - (1-\lambda_{ik}^{s})\ln(\mu_{ik}^{c}) \right| \leqslant \frac{n(n-1)(n-2)\delta}{6}, \\ \lambda_{ij}^{s}, \theta_{ij}^{s} \in [0,1],\ i,j = 1,2,\cdots,n,\ i<j. \end{cases} \tag{M-6}$$

where τ is a predefined acceptable consensus threshold value, δ is a predefined acceptable consistency threshold value, and $\tau,\ \delta \in [0,1]$.

The purpose of building the model (M-6) is to ensure that the adjusted q-ROFPRs satisfy the requirement of acceptable consistency and acceptable consensus. In the model (M-6), the first constraint guarantees that the adjusted q-ROFPRs have an acceptable consensus, the second constraint guarantees that the adjusted q-ROFPRs are acceptable consistent, and the third constraint guarantees that the adjustment parameters $\lambda_{ij}^{s}, \theta_{ij}^{s} \in [0,1]$, where $s = 1,2,\cdots,m$, $i = 1,2,\cdots,n$, $j = 1,2,\cdots,n$ and $i<j$. To obtain the numerical results of the model (M-6), the software package MATLAB is used in this paper.

4.2. A new GDM method with incomplete q-ROFPRs

In the following, we propose a new GDM method, called **Algorithm 2**, based on incomplete q-ROFPRs. Assume that there are m DMs $d_1, d_2, \cdots, d_m$ and assume that there are n alternatives $x_1, x_2, \cdots, x_n$. Let A^s be an incomplete q-ROFPR given by DM d_s, where $s = 1,2,\cdots,m$ and $q \geqslant 1$. Let δ be a predefined threshold value, where $\delta \in [0,1]$, and let τ be a predefined acceptable consensus threshold value, where $\tau \in [0,1]$. The proposed **Algorithm 2** for GDM is now presented as follows:

Algorithm 2:

Step 1: Apply the model (M-2) to determine the unknown q-ROFNs in incomplete q-ROFPRs A^s $(s = 1,2,\cdots,m)$, which are still denoted by q-ROFPRs A^s $(s = 1,2,\cdots,m)$.

Step 2: For those q-ROFPRs which satisfy $MCI(A^s) > \delta$, where δ is a predefined threshold value and $\delta \in [0,1]$, derive their acceptable multiplicative consistent q-ROFPRs $A^s = \left(\alpha_{ij}^{s}\right)_{n\times n}$ $(s = 1,2,\cdots,m)$ by the model (M-4), where $i = 1,2,\cdots,n$, $j = 1,2,\cdots,n$ and $q \geqslant 1$.

Step 3: Based on the q-ROFPRs A^s $(s = 1,2,\cdots,m)$ and Eqs. (10) and (11), compute the weight w_s of DM d_s, where $s = 1,2,\cdots,m$, to get the DMs' weight vector $w = (w_1, w_2, \cdots, w_m)^T$. Based on **Theorem 4.1** and the q-ROFPRs A^s $(s = 1,2,\cdots,m)$, get the collective q-ROFPR A^c.

Step 4: Based on Eq. (12), if $GCI(A^s) \leqslant \tau$ for all $s = 1,2,\cdots,m$, where τ is a predefined acceptable consensus threshold value and $\tau \in [0,1]$, then go to **Step 5**. Otherwise, apply the model (M-6) to adjust each q-ROFPR A^s to reach the acceptable consensus level until $GCI(A^s) \leqslant \tau$ for all $s = 1,2,\cdots,m$, where $\tau \in [0,1]$. Let A'^{s} be the adjusted q-ROFPR for A^s. Aggregate all the adjusted q-ROFPRs into the collective q-ROFPR A'^{c} based on **Theorem 4.1.**

Step 5: Based on the model (M-5) and A'^{c} obtained in **Step 4**, get the q-ROF priority weight $\omega_i = \left(\mu_{\omega_i}, v_{\omega_i}\right)$ of alternative x_i, where $i = 1,2,\cdots,n$.

Step 6: Based on **Definition 2.1**, calculate $S(\omega_i)$ and $H(\omega_i)$ for ω_i, respectively, where $S(\omega_i) = \mu_{\omega_i} - v_{\omega_i}$, $H(\omega_i) = \mu_{\omega_i} + v_{\omega_i}$ and $i = 1,2,\cdots,n$. For any $i,j = 1,2,\cdots,n$ and $i \neq j$, if $S(\omega_i) > S(\omega_j)$, then alternative x_i is better than alternative x_j; if $S(\omega_i) = S(\omega_j)$ and $H(\omega_i) > H(\omega_j)$, then alternative x_i is better than alternative x_j; if $S(\omega_i) = S(\omega_j)$ and $H(\omega_i) = H(\omega_j)$, then alternative x_i and alternative x_j have the same ranking order.

5. Case study and comparisons

5.1. Application example

Example 5.1. An award committee of a conference composed of four Professors d_1, d_2, d_3 and d_4 to grant the best paper award of a conference. There is a set $X = \{x_1$: John's paper, x_2: Mike's paper, x_3: David's paper, x_4: Frank's paper$\}$ of four selected papers to be the candidates of the best paper award of the conference. Let $q = 3$. Suppose that the incomplete q-ROFPRs A^1, A^2, A^3 and A^4 are established by the Professors d_1, d_2, d_3 and d_4, respectively, shown as follows:

$$A^1 = \begin{pmatrix} \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (-,-) & (0.7, 0.5) & (0.5, -) \\ (-,-) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (-,-) & (0.7, 0.1) \\ (0.5, 0.7) & (-,-) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.6, 0.9) \\ (-, 0.5) & (0.1, 0.7) & (0.9, 0.6) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) \end{pmatrix}$$

$$A^2 = \begin{pmatrix} \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.5, 0.8) & (0.7, 0.6) & (-,-) \\ (0.8, 0.5) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.6, -) & (0.4, 0.8) \\ (0.6, 0.7) & (-, 0.6) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.8, 0.7) \\ (-,-) & (0.8, 0.4) & (0.7, 0.8) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) \end{pmatrix}$$

$$A^3 = \begin{pmatrix} \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.7, 0.6) & (-, 0.3) & (0.5, 0.9) \\ (0.6, 0.7) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.8, 0.4) & (0.3, 0.4) \\ (0.3, -) & (0.4, 0.8) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.7, -) \\ (0.9, 0.5) & (0.4, 0.3) & (-, 0.7) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) \end{pmatrix}$$

$$A^4 = \begin{pmatrix} \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (-,-) & (0.9, 0.6) & (0.6, 0.5) \\ (-,-) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.1, 0.7) & (-,-) \\ (0.6, 0.9) & (0.7, 0.1) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.3, 0.6) \\ (0.5, 0.6) & (-,-) & (0.6, 0.3) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) \end{pmatrix}$$

Let δ be the predefined threshold value, where $\delta = 0.1$, and let τ be the given acceptable consensus threshold value, where $\tau = 0.16$. In order to obtain the numerical results, the software package MATLAB is used in this paper.

[Step 1] Four incomplete q-ROFPRs A^1, A^2, A^3 and A^4 are provided by the four Professors d_1, d_2, d_3 and d_4 , respectively, as shown above. For the incomplete q-ROFPRs A^1, A^2, A^3 and A^4, we obtain their complete q-ROFPRs based on the model (M-2), respectively, shown as follows:

$$A^1 = \begin{pmatrix} \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.1332, 0.9992) & (0.7, 0.5) & (0.5, 0.5357) \\ (0.9992, 0.1332) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.988, 0.0941) & (0.7, 0.1) \\ (0.5, 0.7) & (0.9041, 0.988) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.6, 0.9) \\ (0.5357, 0.5) & (0.1, 0.7) & (0.9, 0.6) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) \end{pmatrix}$$

$$A^2 = \begin{pmatrix} \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.5, 0.8) & (0.7, 0.6) & (0.8492, 0.7218) \\ (0.8, 0.5) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.6, 0.8991) & (0.4, 0.8) \\ (0.6, 0.7) & (0.8991, 0.6) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.8, 0.7) \\ (0.7218, 0.8492) & (0.8, 0.4) & (0.7, 0.8) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) \end{pmatrix}$$

$$A^3 = \begin{pmatrix} \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.7, 0.6) & (0.4955, 0.3) & (0.5, 0.9) \\ (0.6, 0.7) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.8, 0.4) & (0.3, 0.4) \\ (0.3, 0.4955) & (0.4, 0.8) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.7, 0.8693) \\ (0.9, 0.5) & (0.4, 0.3) & (0.8693, 0.7) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) \end{pmatrix}$$

$$A^4 = \begin{pmatrix} \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.9997, 0.0927) & (0.9, 0.6) & (0.6, 0.5) \\ (0.0927, 0.9997) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.1, 0.7) & (0.0731, 0.9999) \\ (0.6, 0.9) & (0.7, 0.1) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.3, 0.6) \\ (0.5, 0.6) & (0.9999, 0.0731) & (0.6, 0.3) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) \end{pmatrix}$$

[Step 2] Based on Eq. (4), we get $MCI\left(A^1\right) = 0$, $MCI\left(A^2\right) = 0.7254$, $MCI\left(A^3\right) = 0.6092$ and $MCI\left(A^4\right) = 0.235$. Because $MCI\left(A^2\right) > \delta$, $MCI\left(A^3\right) > \delta$ and $MCI\left(A^4\right) > \delta$, where $MCI\left(A^2\right) = 0.7254$, $MCI\left(A^3\right) = 0.6092$, $MCI\left(A^4\right) = 0.235$ and the predefined threshold value $\delta = 0.1$, we can see that the q-ROFPRs A^2, A^3 and A^4 are unacceptable consistent. Based on the model (M-4), the acceptable multiplicative consistent q-ROFPRs A^2, A^3 and A^4 are acquired as follows:

$$A^2 = \begin{pmatrix} \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.9016, 0.617) & (0.693, 0.6027) & (0.8492, 0.7218) \\ (0.617, 0.9016) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.6, 0.8991) & (0.5593, 0.7974) \\ (0.6027, 0.693) & (0.8991, 0.6) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.7872, 0.723) \\ (0.7218, 0.8492) & (0.7974, 0.5593) & (0.723, 0.7872) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) \end{pmatrix}$$

$$A^3 = \begin{pmatrix} \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.6983, 0.6983) & (0.4955, 0.3) & (0.5605, 0.8059) \\ (0.6983, 0.6983) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.7983, 0.5152) & (0.3, 0.4) \\ (0.3, 0.4955) & (0.5152, 0.7983) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.4823, 0.9611) \\ (0.8059, 0.5605) & (0.4, 0.3) & (0.9611, 0.4823) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) \end{pmatrix}$$

$$A^4 = \begin{pmatrix} \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.9997, 0.0927) & (0.9225, 0.5989) & (0.5995, 0.6304) \\ (0.0927, 0.9997) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.1, 0.7) & (0.0731, 0.9999) \\ (0.5989, 0.9225) & (0.7, 0.1) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.3006, 0.5947) \\ (0.6304, 0.5995) & (0.9999, 0.0731) & (0.5947, 0.3006) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) \end{pmatrix}$$

[Step 3] Based on Eqs. (10) and (11) and the q-ROFPRs A^1, A^2, A^3 and A^4, we get the DMs' weight vector $w = (0.2306, 0.2692, 0.2681, 0.2321)^T$. Based on **Theorem 4.1** and the q-ROFPRs A^1, A^2, A^3 and A^4, we get the collective q-ROFPR A^c, shown as follows:

$$A^c = \begin{pmatrix} \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.5549, 0.4483) & (0.6784, 0.4781) & (0.6201, 0.6726) \\ (0.4483, 0.5549) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.4794, 0.4342) & (0.3108, 0.4328) \\ (0.4781, 0.6784) & (0.4342, 0.4794) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.5186, 0.7844) \\ (0.6726, 0.6201) & (0.4328, 0.3108) & (0.7844, 0.5186) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) \end{pmatrix}$$

[Step 4] Because the predefined acceptable consensus threshold value $\tau = 0.16$, based on Eq. (12), we get $GCI\left(A^1\right) = 0.63 > 0.16$, $GCI\left(A^2\right) = 0.341 > 0.16$, $GCI\left(A^3\right) = 0.2265 > 0.16$ and $GCI\left(A^4\right) = 0.6624 > 0.16$. Then, based on the model (M-6), we get the modified q-ROFPRs A^1, A^2, A^3 and A^4, shown as follows:

$$A^1 = \begin{pmatrix} \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.4747, 0.6424) & (0.7, 0.5) & (0.5, 0.5584) \\ (0.6424, 0.4747) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.6853, 0.3794) & (0.4318, 0.3497) \\ (0.5, 0.7) & (0.3794, 0.6853) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.6, 0.9) \\ (0.5584, 0.5) & (0.3497, 0.4318) & (0.9, 0.6) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) \end{pmatrix}$$

$$A^2 = \begin{pmatrix} \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.6428, 0.617) & (0.693, 0.6027) & (0.8492, 0.7218) \\ (0.617, 0.6428) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.6, 0.5244) & (0.4772, 0.4679) \\ (0.6027, 0.693) & (0.5244, 0.6) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.7872, 0.723) \\ (0.7218, 0.8492) & (0.4679, 0.4772) & (0.723, 0.7872) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) \end{pmatrix}$$

$$A^3 = \begin{pmatrix} \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.6983, 0.6392) & (0.5265, 0.3799) & (0.5605, 0.8059) \\ (0.6392, 0.6983) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.7983, 0.5152) & (0.3, 0.4) \\ (0.3799, 0.5265) & (0.5152, 0.7983) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.4823, 0.9611) \\ (0.8059, 0.5605) & (0.4, 0.3) & (0.9611, 0.4823) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) \end{pmatrix}$$

$$A^4 = \begin{pmatrix} \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.5549, 0.4483) & (0.6784, 0.4781) & (0.5995, 0.6304) \\ (0.4483, 0.5549) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.4795, 0.4342) & (0.3108, 0.4328) \\ (0.4781, 0.6784) & (0.4342, 0.4795) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.5186, 0.7844) \\ (0.6304, 0.5995) & (0.4328, 0.3108) & (0.7844, 0.5186) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) \end{pmatrix}$$

Furthermore, based on **Theorem 4.1**, we get the collective q-ROFPR A^c, shown as follows:

$$A^c = \begin{pmatrix} \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.5862, 0.5803) & (0.6454, 0.4838) & (0.6146, 0.6726) \\ (0.5803, 0.5862) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.6295, 0.4592) & (0.3723, 0.4103) \\ (0.4838, 0.6454) & (0.4592, 0.6295) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.5863, 0.8367) \\ (0.6726, 0.6146) & (0.4103, 0.3723) & (0.8367, 0.5863) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) \end{pmatrix}$$

Based on Eqs. (4) and (12), we obtain $MCI\left(A^1\right) = 0.0333$, $MCI\left(A^2\right) = 0.1$, $MCI\left(A^3\right) = 0.1$, $MCI\left(A^4\right) = 0.0335$, $MCI(A^c) = 0.03$, $GCI\left(A^1\right) = 0.1249$, $GCI\left(A^2\right) = 0.1533$, $GCI\left(A^3\right) = 0.1598$ and $GCI\left(A^4\right) = 0.1011$.

[Step 5] Based on the obtained collective q-ROFPR A^c, we get the q-ROF priority weight $\omega_i = \left(\mu_{\omega_i}, \nu_{\omega_i}\right)$ of alternative x_i, where $i = 1, 2, 3, 4$, $\omega_1 = (0.6582, 0.8941)$, $\omega_2 = (0.6490, 0.8906)$, $\omega_3 = (0.5411, 0.9441)$ and $\omega_4 = (0.6405, 0.8950)$.

[Step 6] Based on the score function S shown in **Definition 2.1**, we get the score values $S(\omega_1)$, $S(\omega_2)$, $S(\omega_3)$ and $S(\omega_4)$ of the q-ROF priority weights ω_1, ω_2, ω_3 and ω_4, respectively, where $\omega_1 = (0.6582, 0.8941)$, $\omega_2 = (0.6490, 0.8906)$, $\omega_3 = (0.5411, 0.9441)$, $\omega_4 = (0.6405, 0.8950)$, $S(\omega_1) = -0.4297$, $S(\omega_2) = -0.4330$, $S(\omega_3) = -0.6832$ and $S(\omega_4) = -0.4542$. Because $S(\omega_1) > S(\omega_2) > S(\omega_4) > S(\omega_3)$, where $S(\omega_1) = -0.4297$, $S(\omega_2) = -0.4330$, $S(\omega_3) = -0.6832$ and $S(\omega_4) = -0.4542$, the ranking order of the alternatives x_1, x_2, x_3 and x_4 is "$x_1 \succ x_2 \succ x_4 \succ x_3$".

5.2. Comparative analysis with two methods presented in [47] and [48]

The GDM methods presented in [47] and [48] only can deal with GDM with complete q-ROFPRs. Thus, it cannot be applied to deal with **Example 5.1**. For the convenience of comparisons, let us consider the GDM problem regarding the rehabilitation programs selection adopted from [48] (Please refer to pages 55–57 of [48] for more details), where three q-ROFPRs R^1, R^2 and R^3 are provided by three DMs d_1, d_2 and d_3, respectively, shown as follows:

$$R^1 = \begin{pmatrix} (\sqrt[3]{0.5}, \sqrt[3]{0.5}) & (0.6, 0.2) & (0.8, 0.6) & (0.3, 0.9) \\ (0.2, 0.6) & (\sqrt[3]{0.5}, \sqrt[3]{0.5}) & (0.5, 0.6) & (0.1, 0.9) \\ (0.6, 0.8) & (0.6, 0.5) & (\sqrt[3]{0.5}, \sqrt[3]{0.5}) & (0.4, 0.8) \\ (0.9, 0.3) & (0.9, 0.1) & (0.8, 0.4) & (\sqrt[3]{0.5}, \sqrt[3]{0.5}) \end{pmatrix},$$

$$R^2 = \begin{pmatrix} (\sqrt[3]{0.5}, \sqrt[3]{0.5}) & (0.7, 0.2) & (0.8, 0.5) & (0.3, 0.6) \\ (0.2, 0.7) & (\sqrt[3]{0.5}, \sqrt[3]{0.5}) & (0.6, 0.8) & (0.2, 0.9) \\ (0.5, 0.8) & (0.8, 0.6) & (\sqrt[3]{0.5}, \sqrt[3]{0.5}) & (0.3, 0.8) \\ (0.6, 0.3) & (0.9, 0.2) & (0.8, 0.3) & (\sqrt[3]{0.5}, \sqrt[3]{0.5}) \end{pmatrix},$$

$$R^3 = \begin{pmatrix} (\sqrt[3]{0.5}, \sqrt[3]{0.5}) & (0.6, 0.4) & (0.7, 0.4) & (0.6, 0.7) \\ (0.4, 0.6) & (\sqrt[3]{0.5}, \sqrt[3]{0.5}) & (0.5, 0.6) & (0.4, 0.9) \\ (0.4, 0.7) & (0.6, 0.5) & (\sqrt[3]{0.5}, \sqrt[3]{0.5}) & (0.5, 0.8) \\ (0.7, 0.6) & (0.9, 0.4) & (0.8, 0.5) & (\sqrt[3]{0.5}, \sqrt[3]{0.5}) \end{pmatrix}.$$

We utilize our proposed **Algorithm 2** to deal with this example given in [48]. Firstly, we investigate the consistency of these three q-ROFPRs R^1, R^2 and R^3 by Eq. (4) of this paper. The consistency indexes $MCI(R^1)$, $MCI(R^2)$ and $MCI(R^3)$ of these three q-ROFPRs R^1, R^2 and R^3, respectively, are $MCI(R^1) = 0.6609$, $MCI(R^2) = 0.3387$ and $MCI(R^3) = 0.2475$. Let $\delta = 0.1$. It can be seen that the q-ROFPRs R^1, R^2 and R^3 do not have the acceptable consistency. Thus, the q-ROFPRs R^1, R^2 and R^3 should be adjusted until they reach the acceptable consistency. By applying the model (M-4), the adjusted q-ROFPRs $R^{1'}$, $R^{2'}$, and $R^{3'}$ of the q-ROFPRs R^1, R^2 and R^3, respectively, are obtained, shown as follows:

$$R^{1'} = \begin{pmatrix} (\sqrt[3]{0.5}, \sqrt[3]{0.5}) & (0.6, 0.2) & (0.8, 0.6) & (0.3, 0.9) \\ (0.2, 0.6) & (\sqrt[3]{0.5}, \sqrt[3]{0.5}) & (0.2847, 0.6036) & (0.1, 0.9) \\ (0.6, 0.8) & (0.6036, 0.2847) & (\sqrt[3]{0.5}, \sqrt[3]{0.5}) & (0.2569, 0.8927) \\ (0.9, 0.3) & (0.9, 0.1) & (0.8927, 0.2569) & (\sqrt[3]{0.5}, \sqrt[3]{0.5}) \end{pmatrix}$$

$$R^{2'} = \begin{pmatrix} (\sqrt[3]{0.5}, \sqrt[3]{0.5}) & (0.6657, 0.2341) & (0.8, 0.5) & (0.3165, 0.5742) \\ (0.2341, 0.6657) & (\sqrt[3]{0.5}, \sqrt[3]{0.5}) & (0.5635, 0.85) & (0.2, 0.9) \\ (0.5, 0.8) & (0.85, 0.5635) & (\sqrt[3]{0.5}, \sqrt[3]{0.5}) & (0.2919, 0.8173) \\ (0.5742, 0.3165) & (0.9, 0.2) & (0.8173, 0.2919) & (\sqrt[3]{0.5}, \sqrt[3]{0.5}) \end{pmatrix}$$

$$R^{3'} = \begin{pmatrix} (\sqrt[3]{0.5}, \sqrt[3]{0.5}) & (0.7249, 0.392) & (0.6951, 0.4138) & (0.6, 0.7) \\ (0.392, 0.7249) & (\sqrt[3]{0.5}, \sqrt[3]{0.5}) & (0.5, 0.6) & (0.4002, 0.8988) \\ (0.4138, 0.6951) & (0.6, 0.5) & (\sqrt[3]{0.5}, \sqrt[3]{0.5}) & (0.4795, 0.8007) \\ (0.7, 0.6) & (0.8988, 0.4002) & (0.8007, 0.4795) & (\sqrt[3]{0.5}, \sqrt[3]{0.5}) \end{pmatrix}$$

Based on Eqs. (10) and (11), we get the DMs' weight vector $w = (0.3330, 0.3348, 0.3322)^T$. Based on **Theorem 4.1**, we get the collective q-ROFPR $R^{c'}$:

$$R^{c'} = \begin{pmatrix} \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.6615, 0.2636) & (0.7635, 0.4989) & (0.3845, 0.7123) \\ (0.2636, 0.6615) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.4314, 0.6756) & (0.1999, 0.8996) \\ (0.4989, 0.7635) & (0.6756, 0.4314) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.3299, 0.8359) \\ (0.7123, 0.3845) & (0.8996, 0.1999) & (0.8359, 0.3299) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) \end{pmatrix}$$

Let the predefined acceptable consensus threshold value $\tau = 0.1$. Based on Eq. (12), we get $GCI\left(R^{1'}\right) = 0.2187 > 0.1$, $GCI\left(R^{2'}\right) = 0.1022 > 0.1$ and $GCI\left(R^{3'}\right) = 0.2175 > 0.1$. It is clear that $GCI\left(R^{1'}\right) = \max_{1 \leqslant s \leqslant 3}[GCI(R^{s'})]$, which means that the q-ROFPR $R^{1'}$ has the lower consensus degree than the other two q-ROFPRs $R^{2'}$ and $R^{3'}$. Hence, we first adjust the q-ROFPR $R^{1'}$ to improve its consensus degree. Based on the model (M-6), we get the adjusted q-ROFPR $R^{1'}$, where

$$R^{1'} = \begin{pmatrix} \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.6, 0.2636) & (0.8, 0.5756) & (0.3845, 0.7288) \\ (0.2636, 0.6) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.4314, 0.6036) & (0.1999, 0.9) \\ (0.5756, 0.8) & (0.6036, 0.4314) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.3299, 0.8927) \\ (0.7288, 0.3845) & (0.9, 0.1999) & (0.8927, 0.3299) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) \end{pmatrix}$$

Based on Eqs. (10) and (11), we get the DMs' weight vector $w = (0.3351, 0.3330, 0.3319)^T$. Based on **Theorem 4.1**, we get the collective q-ROFPR $R^{c'}$, shown as follows:

$$R^{c'} = \begin{pmatrix} \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.6614, 0.2890) & (0.7635, 0.4923) & (0.4178, 0.6642) \\ (0.2890, 0.6614) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.4952, 0.6752) & (0.2518, 0.8996) \\ (0.4923, 0.7635) & (0.6752, 0.4952) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.3586, 0.8361) \\ (0.6642, 0.4178) & (0.8996, 0.2518) & (0.8361, 0.3586) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) \end{pmatrix}$$

Based on Eq. (12), we get $GCI\left(R^{1'}\right) = 0.0998 < 0.1$, $GCI\left(R^{2'}\right) = 0.1268 > 0.1$ and $GCI\left(R^{3'}\right) = 0.1670 > 0.1$, where the predefined threshold value $\delta = 0.1$. It is clear that $GCI\left(R^{3'}\right) = \max_{1 \leqslant s \leqslant 3}[GCI(R^{s'})]$, which means that the q-ROFPR $R^{3'}$ has the lower consensus degree than the other two q-ROFPRs $R^{1'}$ and $R^{2'}$. Hence, we further adjust the q-ROFPR $R^{3'}$ to improve its consensus degree. We get the adjusted q-ROFPR $R^{3'}$ via the model (M-6), where

$$R^{3'} = \begin{pmatrix} \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.7230, 0.3546) & (0.6951, 0.4380) & (0.4278, 0.7) \\ (0.3546, 0.7230) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.5, 0.6117) & (0.2694, 0.8988) \\ (0.4380, 0.6951) & (0.6117, 0.5) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.3585, 0.8007) \\ (0.7, 0.4278) & (0.8988, 0.2694) & (0.8007, 0.3585) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) \end{pmatrix}$$

Based on Eqs. (10) and (11), we get the DMs' weight vector $w = (0.3338, 0.3328, 0.3335)^T$. Based on **Theorem 4.1**, we get the collective q-ROFPR $R^{c'}$, shown as follows:

$$R^{c'} = \begin{pmatrix} \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.6609, 0.2797) & (0.7634, 0.5014) & (0.3735, 0.6642) \\ (0.2797, 0.6609) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.4953, 0.6794) & (0.2209, 0.8996) \\ (0.5014, 0.7634) & (0.6794, 0.4953) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.3256, 0.8360) \\ (0.6642, 0.3735) & (0.8996, 0.2209) & (0.8360, 0.3256) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) \end{pmatrix}$$

Based on Eq. (4) and Eq. (12), we get $MCI\left(R^{1'}\right) = 0.1$, $MCI\left(R^{2'}\right) = 0.1$, $MCI\left(R^{3'}\right) = 0.1$, $MCI(R^{c'}) = 0.0884$, $GCI\left(R^{1'}\right) = 0.0748$, $GCI\left(R^{2'}\right) = 0.0942$ and $GCI\left(R^{3'}\right) = 0.0998$. By plugging the collective q-ROFPR $R^{c'}$ into the model (M-

5), we get the q-ROF priority weights ω_1, ω_2, ω_3 and ω_4, where $\omega_1 = (0.5823, 0.9247)$, $\omega_2 = (0.3180, 0.9892)$, $\omega_3 = (0.4724, 0.9636)$ and $\omega_4 = (0.8676, 0.6945)$. Based on the score function S shown in **Definition 2.1**, we get the score values $S(\omega_1)$, $S(\omega_2)$, $S(\omega_3)$ and $S(\omega_4)$ of the q-ROF priority weights ω_1, ω_2, ω_3 and ω_4, respectively, where $\omega_1 = (0.5823, 0.9247)$, $\omega_2 = (0.3180, 0.9892)$, $\omega_3 = (0.4724, 0.9636)$, $\omega_4 = (0.8676, 0.6945)$, $S(\omega_1) = -0.5932$, $S(\omega_2) = -0.9357$, $S(\omega_3) = -0.7892$ and $S(\omega_4) = 0.3181$. Because $S(\omega_4) > S(\omega_1) > S(\omega_3) > S(\omega_2)$, where $S(\omega_1) = -0.5932$, $S(\omega_2) = -0.9357$, $S(\omega_3) = -0.7892$ and $S(\omega_4) = 0.3181$, the ranking order of the alternatives x_1, x_2, x_3 and x_4 is "$x_4 \succ x_1 \succ x_3 \succ x_2$". The ranking order "$x_4 \succ x_1 \succ x_3 \succ x_2$" of the alternatives obtained by the proposed **Algorithm 2** is the same as the ones obtained by the GDM methods presented in Zhang et al. [47] and [48].

By comparing with the GDM methods presented in Zhang et al. [47] and [48], the proposed **Algorithm 2** has the following advantages:

(1) The GDM methods presented in Zhang et al. [47] and [48] do not have the capability to deal with incomplete q-ROFPRs, whereas the proposed **Algorithm 2** can deal with incomplete q-ROFPRs for GDM. The decision making information in the proposed **Algorithm 2** includes not only the complete information of DMs, but also the incomplete preference information, which enhances the integrity of decision making information.

(2) In Zhang et al. [47], DMs' weights are set in advance, which seems to be unreasonable and ignores different importance of different DMs. Moreover, there is no rule regarding how to set these DMs' weights in Zhang et al. [47]. In the proposed **Algorithm 2**, after performing the consensus analysis, we offer a model to derive DMs' weights. Moreover, DMs' weights in Zhang et al. [47] are fixed and unchanged in the whole GDM process. By contrast, in the proposed **Algorithm 2**, the weight vector of DMs varies with different q-ROFPRs. Therefore, the weight vector obtained by the proposed **Algorithm 2** is more flexible, which reflects the importance of different q-ROFPRs established by different DMs to match actual GDM processes.

(3) The consistency levels of preference relations have great influences on the derived priority vector and the ranking order of alternatives. The higher the inconsistency level of a preference relation is, the more the irrationality of the ranking order of alternatives is. Therefore, to ensure reasonable ranking orders of alternatives, consistency levels of preference relations should be kept within a certain range. When a consistency level of a preference relation is beyond an acceptable range, the preference relation should be adjusted into a new one with an acceptable consistency. However, the GDM method presented in Zhang et al. [47] does not consider the consistency adjustment. By contrast, the proposed **Algorithm 2** proposes an effective technique to adjust an unacceptable consistent q-ROFPR into an acceptable consistent q-ROFPR. Taking the aforementioned three q-ROFPRs R^1, R^2 and R^3 as an example, based on Eq. (4), we get $MCI\left(R^1\right) = 0.6609$, $MCI\left(R^2\right) = 0.3387$ and $MCI\left(R^3\right) = 0.2475$. Let the predefined threshold value $\delta = 0.1$. Because $MCI\left(R^1\right) > 0.1$, $MCI\left(R^2\right) > 0.1$ and $MCI\left(R^3\right) > 0.1$, where $MCI\left(R^1\right) = 0.6609$, $MCI\left(R^2\right) = 0.3387$ and $MCI\left(R^3\right) = 0.2475$, it is concluded that the q-ROFPRs R^1, R^2, and R^3 do not have an acceptable consistency. The GDM method presented in Zhang et al. [47] obtains the priority vector of alternatives directly from the three q-ROFPRs R^1, R^2 and R^3 by using a programming model without the consideration of the consistency adjustment. The proposed **Algorithm 2** has the advantage that it has the capability to deal with unacceptable consistent q-ROFPRs and provides an adjustment process to improve the consistency of unacceptable consistent q-ROFPRs.

(4) While examining and improving the consistency of a q-ROFPR, Zhang *et al.*'s method [48] needs to establish a multiplicative consistent q-ROFPR besides the original q-ROFPR itself, whereas the proposed **Algorithm 2** has the advantage that it can complete this process only based on the initial q-ROFPR. Moreover, to improve the consistency of a q-ROFPR, Zhang *et al.*'s method [48] may need several iteration times, whereas the proposed **Algorithm 2** has the advantage that it improves the consistency of a q-ROFPR only by solving a model.

(5) To improve the consistency of a q-ROFPR, all of the preference information in the q-ROFPR will be approached to its consistent q-ROFPR at the same time. It means that once a q-ROFPR does not meet the consistency requirement, all of the preference information in the q-ROFPR will be adjusted to close to the ones of its consistent q-ROFPR, which may make the original preference information to lose its original characteristics. For example, the adjusted q-ROFPRs $R^{1''}$, $R^{2''}$, and $R^{3''}$ of the q-ROFPRs R^1, R^2 and R^3 obtained by Zhang et al.'s method [48], respectively, are as follows:

$$R^{1''} = \begin{pmatrix} \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.6000, 0.3127) & (0.5377, 0.4241) & (0.2190, 0.7505) \\ (0.3127, 0.6000) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.3765, 0.5698) & (0.1274, 0.8378) \\ (0.4241, 0.5377) & (0.5698, 0.3765) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) & (0.1841, 0.8000) \\ (0.7505, 0.2190) & (0.8378, 0.1274) & (0.8000, 0.1841) & \left(\sqrt[3]{0.5}, \sqrt[3]{0.5}\right) \end{pmatrix}$$

$$R^{2''} = \begin{pmatrix} (\sqrt[3]{0.5}, \sqrt[3]{0.5}) & (0.7044, 0.2515) & (0.6625, 0.3028) & (0.3244, 0.6454) \\ (0.2515, 0.7044) & (\sqrt[3]{0.5}, \sqrt[3]{0.5}) & (0.4331, 0.5544) & (0.1489, 0.8298) \\ (0.3028, 0.6625) & (0.5544, 0.4331) & (\sqrt[3]{0.5}, \sqrt[3]{0.5}) & (0.1837, 0.7999) \\ (0.6454, 0.3244) & (0.8298, 0.1489) & (0.7999, 0.1837) & (\sqrt[3]{0.5}, \sqrt[3]{0.5}) \end{pmatrix}$$

$$R^{3''} = \begin{pmatrix} (\sqrt[3]{0.5}, \sqrt[3]{0.5}) & (0.6000, 0.4000) & (0.6000, 0.4000) & (0.3877, 0.6123) \\ (0.4000, 0.6000) & (\sqrt[3]{0.5}, \sqrt[3]{0.5}) & (0.5000, 0.5000) & (0.2968, 0.7032) \\ (0.4000, 0.6000) & (0.5000, 0.5000) & (\sqrt[3]{0.5}, \sqrt[3]{0.5}) & (0.2968, 0.7032) \\ (0.6123, 0.3877) & (0.7032, 0.2968) & (0.7032, 0.2968) & (\sqrt[3]{0.5}, \sqrt[3]{0.5}) \end{pmatrix}$$

In order to further compare the performances of different GDM methods, we propose a distance measure $D\left(A^1, A^2\right)$ between two q-ROFPRs A^1 and A^2, shown as follows:

$$D\left(A^1, A^2\right) = \frac{1}{n(n-1)} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(\left| \ln\left(\mu_{ij}^1\right) - \ln\left(\mu_{ij}^2\right) \right| + \left| \ln\left(v_{ij}^1\right) - \ln\left(v_{ij}^2\right) \right| \right) \tag{18}$$

where $A^1 = \left(\alpha_{ij}^1\right)_{n\times n}$, $\alpha_{ij}^1 = \left(\mu_{ij}^1, v_{ij}^1\right)$ is a q-ROFN, $A^2 = \left(\alpha_{ij}^2\right)_{n\times n}$, $\alpha_{ij}^2 = \left(\mu_{ij}^2, v_{ij}^2\right)$ is a q-ROFN, $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, n$ and $q \geqslant 1$. Based on Eq. (18), we obtain the distance $D\left(R^k, R^{k''}\right)$ between the original q-ROFPRs R^k ($k = 1, 2, 3$) and the adjusted q-ROFPRs $R^{k''}$ ($k = 1, 2, 3$) obtained by Zhang *et al.*'s method [48], where $D\left(R^1, R^{1''}\right) = 0.2594$, $D\left(R^2, R^{2''}\right) = 0.2197$ and $D\left(R^3, R^{3''}\right) = 0.1752$. In the same way, based on Eq. (18), we get the distance $D\left(R^k, R^{k'}\right)$ between the original q-ROFPRs R^k ($k = 1, 2, 3$) and the adjusted q-ROFPRs $R^{k'}$ ($k = 1, 2, 3$) obtained by the proposed **Algorithm 2**, where $D\left(R^1, R^{1'}\right) = 0.1605$, $D\left(R^2, R^{2'}\right) = 0.0398$ and $D\left(R^3, R^{3'}\right) = 0.1244$. By comparing the adjusted q-ROFPRs obtained by the proposed **Algorithm 2** with the adjusted q-ROFPRs obtained by Zhang *et al.*'s GDM method [48], it can be seen that the adjusted q-ROFPRs obtained by the proposed **Algorithm 2** have the less deviation from the original q-ROFPRs due to the fact that $D\left(R^k, R^{k'}\right) < D\left(R^k, R^{k''}\right)$ for all $k = 1, 2, 3$. Obviously, the proposed **Algorithm 2** has the advantage that it possesses the smallest information loss compared to Zhang *et al.*'s method [48].

(6) The GDM methods presented in Zhang et al. [47] and [48] ignored the consensus reaching processes. To achieve the consensus of q-ROFPRs has been recognized as one of essence goals in GDM [31]. The proposed **Algorithm 2** has the advantage that it develops a model to improve the consensus of q-ROFPRs, which can ensure the smallest information loss to reach an acceptable consistency of the adjusted q-ROFPRs. Based on Eq. (12), the consensus indices $GCI\left(R^1\right)$, $GCI\left(R^2\right)$ and $GCI\left(R^3\right)$ of the original q-ROFPRs R^1, R^2 and R^3, respectively, used in Zhang *et al.*'s method [47] are $GCI\left(R^1\right) = 0.1533$, $GCI\left(R^2\right) = 0.1155$ and $GCI\left(R^3\right) = 0.2008$. Based on Eq. (12), the consensus indices $GCI\left(R^{1'}\right)$, $GCI\left(R^{2'}\right)$ and $GCI\left(R^{3'}\right)$ of the adjusted q-ROFPRs $R^{1'}$, $R^{2'}$ and $R^{3'}$, respectively, obtained by Zhang *et al.*'s method [48] are $GCI\left(R^{1'}\right) = 0.1273$, $GCI\left(R^{2'}\right) = 0.1003$ and $GCI\left(R^{3'}\right) = 0.1631$. The consensus indices $GCI\left(R^1\right)$, $GCI\left(R^2\right)$ and $GCI\left(R^3\right)$ of the adjusted q-ROFPRs R^1, R^2 and R^3, respectively, obtained by the proposed **Algorithm 2** are $GCI\left(R^1\right) = 0.0748$, $GCI\left(R^2\right) = 0.0942$ and $GCI\left(R^3\right) = 0.0998$. Obviously, the adjusted q-ROFPRs obtained by the proposed **Algorithm 2** get higher consensus degrees than the ones of the initial q-ROFPRs used in Zhang *et al.*'s method [47] and the ones of the adjusted q-ROFPRs obtained by Zhang *et al.*'s method [48].

6. Conclusions

In this paper, we have proposed a new group decision making (GDM) method based on the multiplicative consistency-and-consensus preference analysis for incomplete q-rung orthopair fuzzy preference relations (q-ROFPRs). Firstly, we present the multiplicative consistency concept for q-ROFPRs. Following the proposed multiplicative consistency concept of q-ROFPRs, we propose a consistency index for a q-ROFPR. Then, the definition of an acceptable consistent q-ROFPR is defined. A model for estimating unknown values in an incomplete q-ROFPR is proposed through minimizing the consistency index.

Meanwhile, we propose an optimization-based algorithm to get the acceptable multiplicative consistent q-ROFPR when a q-ROFPR is inconsistent. Subsequently, we propose a new individual decision making method with an incomplete q-ROFPR. Afterwards, we propose a consensus measurement method for gauging the consensus degree of decision makers in GDM processes. Furthermore, we propose a consensus improvement method to complete the consensus reaching process. When all experts reach a consensus level, a model is set up to obtain the priority vector of each expert, and then the ranking order of alternatives can be obtained. Afterward, we offer a GDM method based on the preference analysis for incomplete q-ROFPRs. Finally, we apply the proposed GDM method to deal with an application example and make some comparative analyses with Zhang *et al.*'s method [47] and Zhang *et al.*'s method [48]. The proposed GDM method outperforms Zhang *et al.*'s method [47] and Zhang *et al.*'s method [48] for GDM in incomplete q-rung orthopair fuzzy environments. It is worth of future research to propose new multiattribute decision making methods based on [1,2,3,4,8,18,25,28,32,40,43].

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